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DIRICHLET PROBLEMS FOR SINGULAR ELLIPTIC EQUATIONS

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ABSTRACT. Boundary value problems are formulated for the equation

$$(*) \quad L[u] = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n-1} b_i \frac{\partial u}{\partial x_i} + \frac{h}{x_n} \frac{\partial u}{\partial x_n} + cu = f$$

in a bounded domain G in E_n with boundary $\partial G = S_1 \cup S_2$ where S_1 is in $x_n=0$ and S_2 is in $x_n>0$. A uniqueness theorem is established for (*) when boundary data is only given on S_2 for

$$h(x_1, \cdots, x_{n-1}, 0) \geqq 1;$$

whereas an existence and uniqueness theorem for the Dirichlet problem is proved for $h(x_1, x_2, \cdots, x_{n-1}, 0) < 1$.

Let G be a bounded domain in a half space $x_n>0$ with boundary $\partial G = S_1 \cup S_2$ where S_1 is contained in the hyperplane $x_n=0$ and S_2 lies entirely in $x_n>0$. We consider a class of second order linear partial differential equations

$$(1) \qquad L[u] = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n-1} b_i \frac{\partial u}{\partial x_i} + \frac{h}{x_n} \frac{\partial u}{\partial x_n} + cu = f$$

where the coefficients $a_{ij}(\bar{x})$, $b_i(\bar{x})$, $h(\bar{x})$, $c(\bar{x})$ and $f(\bar{x})$ are continuous functions of $\bar{x} = (x_1, x_2, \cdots, x_n)$ in $\bar{G}$, and $c(\bar{x}) \leqq 0$ in $\bar{G}$. Furthermore, $L[u]$ is uniformly elliptic in $\bar{G}$, i.e. there exists a positive constant m such that

$$(2) \qquad \sum_{i,j=1}^{n} a_{ij}(\bar{x}) \xi_i \xi_j \geqq m \sum_{i=1}^{n} \xi_i^2$$

for all n-tuples of real numbers $(\xi_1, \cdots, \xi_n)$ and all $\bar{x} \in \bar{G}$. By normalization, $a_{nn}(\bar{x}) = 1$.

In this paper we shall formulate some boundary value problems of (1) in $\bar{G}$. Since the coefficient of $\partial u / \partial x_n$ goes to infinity as $x_n \to 0$, it is obvious

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that the behavior of $h(\bar{x})$ near $x_n=0$ plays an important part in our formulation. A well-known example of (1) is given by the generalized axially symmetric potential equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \frac{k}{x_n}\frac{\partial u}{\partial x_n} = 0 \tag{3}$$

where k is a real number.

For $k<1$, P. Brousse and H. Poncin have shown that the Dirichlet problem can be solved if smooth boundary is admitted [1]. However, for $k\geqq 1$, the Dirichlet problem is in general not solvable. A. Huber has shown that there exists a unique function u in $\bar{G}$, which satisfies (3) and assumes bounded continuous data on S_2 ([3], [4]). Accordingly, we shall formulate our boundary value problems, based on the assumption whether $h(x_1, \cdots, x_{n-1}, 0)<1$ or $h(x_1, \cdots, x_{n-1}, 0)\geqq 1$.

THEOREM 1. *Let $h(\bar{x})$ be a C^2-function in $\bar{G}$, even in x_n, and $h(x_1, x_2, \cdots, x_{n-1}, 0)\geqq 1$. Then there is at most one solution $u(\bar{x})$ of* (1) *which is regular in G, remains bounded when $x_n\to 0$, and assumes given continuous values $\Phi(\bar{x})$ on the boundary S_2.*

PROOF. It is sufficient to prove that any bounded and regular solution $u(\bar{x})$ of $L[u]=0$ which vanishes on S_2 must vanish identically in G. We want to construct a barrier function $w(\bar{x})$ with the following properties: (a) $w(\bar{x})$ is positive in $\bar{G}$, (b) $w(\bar{x})$ converges uniformly to infinity when $x_n\to 0$ and (c) $L[w]<0$.

If such a function $w(\bar{x})$ exists, then, for any $\varepsilon > 0$, we form $v(\bar{x})= \varepsilon w(\bar{x})+u(\bar{x})$. Let $P\in G$. Then, from property (b) of $w(\bar{x})$, there exists an η such that

$$v(\bar{x}) \geqq 0 \quad \text{for } 0 < x_n \leqq \eta.$$

We require η is so small that $P\in G_\eta=G\cap D_\eta$ where $D_\eta=\{\bar{x}|x_n>\eta\}$. This implies $v(\bar{x})\geqq 0$ on ∂G_η, and, from property (c) of $w(\bar{x})$, $L[v]<0$. Hence, by maximum principle for elliptic operators [2], $v(\bar{x})\geqq 0$ in G_η or $u(P)\geqq -\varepsilon w(P)$. Similarly, by defining $v(\bar{x})=\varepsilon w(\bar{x})-u(\bar{x})$, we obtain $u(P)\leqq \varepsilon w(P)$. Hence $|u(P)|\leqq \varepsilon w(P)$. Since ε is arbitrary, $u(P)=0$.

The function $w(\bar{x})$ can be defined as

$$w(\bar{x}) = -\ln x_n - (x_1 - \alpha)^\beta + k \tag{4}$$

where α is chosen such that $x_1-\alpha>1$ for all $\bar{x}=(x_1, \cdots, x_n)\in G$, whereas β and k are positive constants to be determined later.

Then

$$\begin{aligned} L[w] = & -a_{11}(\bar{x})\beta(\beta - 1)(x_1 - \alpha)^{\beta-2} - b_1(\bar{x})\beta(x_1 - \alpha)^{\beta-1} \\ & + (1 - h(\bar{x}))/x_n^2 + c(\bar{x})w(\bar{x}). \end{aligned} \tag{5}$$

Applying Taylor's theorem with remainder to the function $F(x_n)=h(\xi_1,\cdots,\xi_{n-1},x_n)$ where ξ_i, $i=1,\cdots,n-1$, are considered as parameters and $(\xi_1,\cdots\xi_n,x_n)\in G$, we obtain

$$F(x_n)=F(0)+(F'(0)/1!)x_n+(F''(\xi_n)/2!)x_n^2$$

where $0<\xi_n<x_n$.

Then, from the hypothesis on $h(\bar{x})$, we have

$$\begin{aligned}\frac{1-h(\xi_1,\cdots,\xi_{n-1},x_n)}{x_n^2} &= \frac{1-h(\xi_1,\cdots,\xi_{n-1},0)-h(\xi_1,\cdots,\xi_{n-1},0)}{x_n^2} - \frac{h_{x_nx_n}(\xi_1,\cdots,\xi_{n-1},\xi_n)}{x_n^2}\\ &< -h_{x_nx_n}(\xi_1,\cdots,\xi_{n-1},\xi_n)\leqq A\end{aligned}\tag{6}$$

where A is chosen to be a positive constant greater than $\max_{\bar{x}\in\bar{G}}|h_{x_nx_n}(\bar{x})|$.

Moreover,

$$a_{11}(\bar{x})>m \quad \text{by (2).}\tag{7}$$

Hence, combining (5), (6) and (7), we have

$$L[w]<-m\beta(\beta-1)(x_1-\alpha)^{\beta-2}-\beta b_1(\bar{x})(x_1-\alpha)^{\beta-1}+A+C(\bar{x})w(\bar{x}).$$

Choose β such that $m(\beta-1)>\max(3B,2)$ and $\beta(\beta-1)>3A/m$ where $B=\max_{\bar{x}\in\bar{G}}|b_1(\bar{x})(x_1-\alpha)|$.

The number k is now chosen such that the equation $w(\bar{x})$ is positive everywhere in $\bar{G}$. Then,

$$\begin{aligned}L[w]&<-m\beta(\beta-1)+\tfrac{1}{3}m\beta(\beta-1)+A+c(\bar{x})w(\bar{x})\\ &<-(2m\beta/3)(\beta-1)+A<-m\beta(\beta-1)/3<0.\end{aligned}$$

In order to show that the Dirichlet problem is solvable when $h(x_1,\cdots,x_{n-1},0)<1$, we plan to apply the Schauder theorem on the regular elliptic differential equations.

For $0<\alpha<1$, let $C^{m+\alpha}(\bar{\mathscr{D}})$ be the set of functions $u\in C^m(\bar{\mathscr{D}})$ whose mth order derivatives satisfy a Hölder condition in $(\bar{\mathscr{D}})$ with exponent α. For $u\in C^{m+\alpha}(\bar{\mathscr{D}})$, define

$$\|u\|_{m+\alpha}^{\mathscr{D}}=\max_{x\in\bar{\mathscr{D}}}\sum_{k=0}^{m}|\mathscr{D}^k u|+\max_{P,Q\in\bar{\mathscr{D}}}\frac{|D^m u(P)-D^m u(\mathcal{O})|}{|P-\mathcal{O}|^{\alpha}}.$$

Then the Schauder result can be stated as follows. (See **[5]**, **[6]**.)

LEMMA (SCHAUDER). *Let $\mathscr{D}$ be a bounded domain with boundary $\partial\mathscr{D}$ of class $C^{2+\alpha}$. Let*

$$\mathscr{L}(u) = \sum_{i,j=1} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i-1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = f \tag{8}$$

be a uniformly elliptic differential equation in $\mathscr{D}$. Assume that the coefficients a_{ij}, b_i, c, f are in $C^\alpha(\overline{\mathscr{D}})$ satisfying $|a_{ij}|$, $|b_i|$, $|c|$, $|f| \leqq K$, a constant and $c \leqq 0$ in $\overline{\mathscr{D}}$.

If ϕ is any continuous function on $\partial\mathscr{D}$, there exists a unique solution $u \in C^{2+\alpha}(\mathscr{D}) \cap C^0(\overline{\mathscr{D}})$ of (8) with $u=\phi$ on $\partial\mathscr{D}$.

Accordingly, we formulate the following existence theorem:

THEOREM 2. *Let G be a bounded domain in $x_n>0$ with boundary $\partial G = S_1 \cup S_2$ of class $C^{2+\alpha}$ where S_1 is a subset of $x_n=0$ and S_2 in $x_n>0$. Let the coefficients a_{ij}, k_i, h, c and f be in $C^\alpha(\bar{G})$ satisfying $|a_{ij}|$, $|b_i|$, $|h|$, $|f| \leqq K$, and $C \leqq 0$ in $\bar{G}$.*

If ϕ is any continuous function on ∂G, there exists a unique solution $u \in C^{2+\alpha}(G) \cap C^0(\bar{G})$ of (1) with $u=\phi$ on ∂G.

PROOF. The uniqueness of the solution can be handled easily by Hopf's maximum principle.

Extend the function ϕ continuously to the closed domain $\bar{G}$ and denote $M=\max_{\bar{x}\in\bar{G}} \phi(\bar{x})$. Let $\eta>0$, and denote $G_\eta = G \cap E\{x_n > \eta\}$. According to Schauder's lemma, there exists a solution $u_\eta \in C^{2+\alpha}(G_\eta) \cap C^0(\bar{G}_\eta)$ of (1) with $u_\eta = \phi$ on ∂G_η. Extend the function u_η continuously on the whole domain $\bar{G}$ by defining $u_\eta=\phi$ on $\bar{G} \backslash \bar{G}_\eta$. In this way we define a family U of functions $\{u_\eta(\bar{x})\}$, in $\bar{G}$ corresponding to all values of η, $0<\eta \leqq \max\{x_n | \bar{x}=(x_1 \cdots x_n) \in \bar{G}\}$.

Let η^* be a fixed, but arbitrary small number. The family U is uniformly bounded by M, and from Schauder's estimate

$$\|u\|_{2+\alpha}^{G_\eta} \leqq \tilde{K}(\|f\|_\alpha^{G_\eta} + \|u\|_0^{G_\eta}) \quad ([5], [6]) \tag{9}$$

this family is also equicontinuous in G_η. Then, by Arzela's lemma, it is possible to select a sequence $\{U_i^{(1)}(x)\} = \{u_{\eta_i}(\bar{x})\}$ so that $U_i^{(1)}$, together with its first and second derivatives, converges uniformly in G_η. The sequence $\{U_i^{(1)}(x)\}$ is chosen in the order that $i>j$ if $\eta_i < \eta_j$. Hence, for $\eta_i < \eta$, all $\{U_i^{(1)}(\bar{x})\}$ are solutions of (1) in G_{η^*}.

For each j, let $\eta_j^* = \eta^*/j$. Since $\{U_i^{(1)}(\bar{x})\}$ is also a family of uniform boundedness and equicontinuity in $G_{\eta_2^*}$, we can select a uniformly convergent subsequence $\{U_i^{(2)}(\bar{x})\}$ in $G_{\eta_2^*}$ and so forth.

Now, we choose from the sequences

$$U_1^{(1)}, U_2^{(1)}, \cdots, U_i^{(1)}, \cdots,$$
$$U_1^{(2)}, U_2^{(2)}, \cdots, U_i^{(2)}, \cdots$$

the "diagonal" subsequence $\{U_n^{(n)}(\bar{x})\}$ which converges uniformly to a regular solution $u(\bar{x})$ of (1) in G and assumes $\phi(\bar{x})$ on S_2. We still have to prove that for any $Q \in S_1$, $u(\bar{x})$ is continuous and $u(Q)=\phi(Q)$.

Let $Q=\bar{x}^0=(x_1^0, x_2^0, \cdots, x_{n-1}^0, 0) \in S_1$. We want to construct a barrier function $v(\bar{x})$ with the following properties (a) it is continuous in a sufficiently small neighborhood $w_Q=\{\bar{x} \mid |\bar{x}-\bar{x}^0|<\rho,\ x_n \geqq 0\}$, (b) it vanishes at Q, (c) it is positive in $w_G \backslash Q$ and (d) $L[v]<-1$.

The function $v(\bar{x})$ can be defined as

$$v(x_1, \cdots, x_{n-1}, x_n) = x_n^\beta + \sum_{i=1}^{n} (x_i - x_i^0)^2 \tag{10}$$

where $0<\beta<1$.

It is clear that $v(\bar{x})$ satisfies properties (a), (b) and (c). However, we still have to determine the exact value of β such that $L[v]<-1$.

By elementary computations,

$$\begin{aligned} L[v] &= 2(a_{11}(\bar{x}) + \cdots + a_{n-1,n-1}(\bar{x})) + \beta(\beta-1)x_n^{\beta-2} \\ &\quad + 2[b_1(\bar{x})(x_1 - x_1^0) + \cdots + h_{n-1}(\bar{x})(x_{n-1} - x_{n-1}^0)] \\ &\quad + \beta h(\bar{x})x_n^{\beta-2} + c(\bar{x})v(\bar{x}) \\ &< 2(n-1)K + 2\rho(n-1)K + \beta[\beta - 1 + h(\bar{x})]x_n^{\beta-2}. \end{aligned}$$

Choose β such that $0<\beta<1-h(x_1, \cdots, x_{n-1}, 0)$. Then, by continuity, and for sufficiently small values of ρ, we have $L[v]<-1$.

Since $\phi(\bar{x})$ is continuous, for any $\varepsilon>0$, we can find a semicircular neighborhood $w'_Q \subset w_Q$ such that

$$\phi(Q) - \varepsilon \leqq \phi(P) \leqq \phi(Q) + \varepsilon.$$

Consider the functions

$$A(P) = \phi(Q) - \varepsilon - kv(P) \quad \text{and} \quad B(P) = \phi(Q) + \varepsilon + kv(P).$$

For a sufficiently large positive number k, we have $B(P)>M>A(P)$ on the semicircular part of the boundary w'_Q and $L[A]>K$ and $L[B]<-K$ everywhere in w'_Q. Choose a nonempty domain $w_q=w'_Q \cap E\{x_n>q>0\}$. On ∂w_q, $B(P)>M \geqq U_n^n(P)$, where U_n^n is an arbitrary member of the sequence $\{U_n^n\}$ and satisfies (1) in w_q, whereas in w_q, we have $L[B-U_n^n]= L[B]-L[U_n^n]<-K+f(\bar{x}) \leqq 0$. Then, by maximum principle, $U_n^n(P) \leqq B(P)$ in w_q. Similarly, $U_n^n(P) \geqq A(P)$ in w_q. Thus we have $A(P) \leqq U_n^n(P) \leqq B(P)$. Let $n \to \infty$;

$$A(P) \leqq u(P) \leqq B(P);$$

and then let $P \to Q$;

$$f(Q) - \varepsilon \leqq \lim_{P \to Q} u(P) \leqq f(Q) + \varepsilon$$

or

$$\lim_{P \to Q} u(P) = f(Q).$$

REMARK. M. Schechter [5] has established an existence theorem of Dirichlet problem of $Lu=f$, similar to Theorem 2 by means of Schauder's lemma, but the conditions on the coefficients of the equation are different from ours and his proof does not make use of the construction of barrier function.

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