A CLASS OF UNIVALENT FUNCTIONS

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ABSTRACT. A sharp coefficient estimate is obtained for a class $D(\alpha)$ of functions univalent in the open unit disc. The radius of convexity and an arclength result are also determined for the class.

Let $D(\alpha)$ denote the class of functions $f(z)=z+a_2z^2+\cdots$ analytic in the open unit disc E and satisfying

$$(1) \qquad |(f'(z) - 1)/(f'(z) + 1)| < \alpha, \qquad z \in E,$$

for some α, $0<\alpha\leq1$. The values $f'(z)$ lie inside the circle in the right half plane with center $(1+\alpha^2)/(1-\alpha^2)$ and radius $2\alpha/(1-\alpha^2)$. The class $D(\alpha)$ is a subclass of the class of functions whose derivative has positive real part and hence a function in $D(\alpha)$ is univalent in E. If $f \in D(\alpha)$ it follows from Schwarz lemma that $f'(z)=(1-\alpha z\theta(z))/(1+\alpha z\theta(z))$, where $\theta(z)$ is analytic and $|\theta(z)|\leq1$ in E.

A class of starlike functions has been studied by Padmanabhan [5] in which $f'(z)$ is replaced by $zf'(z)/f(z)$ in inequality (1).

A sharp coefficient estimate for the class $D(\alpha)$ is proved in Theorem 1 using a technique of Clunie and Keogh [2]. In Theorem 2 the radius of convexity of the class is obtained and in Theorem 3 an arclength result is given.

THEOREM 1. *If $f(z)=z+\sum_{n=2}^{\infty} a_nz^n$ is in $D(\alpha)$ for some α, $0<\alpha\leq1$, then $|a_n|\leq2\alpha/n$, $n=2, 3, \cdots$. The inequality is sharp.*

PROOF. Since $f(z)$ is in $D(\alpha)$, then $f'(z)=(1+\alpha z\theta(z))/(1-\alpha z\theta(z))$, where $\theta(z)=\sum_{n=1}^{\infty} t_nz^n$ is analytic and $|\theta(z)|\leq1$ for $z \in E$. Then

$$f'(z) - 1 = \alpha z\theta(z)\{f'(z) + 1\},$$

or

$$(2) \qquad \sum_{n=2}^{\infty} na_nz^{n-1} = \alpha\left(\sum_{n=0}^{\infty} t_nz^n\right)\left(2z + \sum_{n=2}^{\infty} na_nz^n\right).$$

Equating corresponding coefficients in (2) gives

$$na_n = \alpha\{(n - 1)t_0a_{n-1} + (n - 2)t_1a_{n-2} + \cdots + 2t_{n-3}a_2 + 2t_{n-2}\}.$$

Thus a_n depends only on $a_2, a_3, \cdots, a_{n-1}$ and θ for $n \geq 2$. Hence, for $n \geq 2$, it follows from (2) that

$$\sum_{k=2}^{n} k a_k z^{k-1} + \sum_{k=n+1}^{\infty} b_k z^{k-1} = \alpha \theta(z) \left\{ 2z + \sum_{k=2}^{n-1} k a_k z^k \right\},$$

which yields

$$\left| \sum_{k=2}^{n} k a_k z^{k-1} + \sum_{k=n+1}^{\infty} b_k z^{k-1} \right|^2 = \alpha^2 |\theta(z)|^2 \left| 2z + \sum_{k=2}^{n-1} k a_k z^k \right|^2$$

$$\leq \alpha^2 \left| 2z + \sum_{k=2}^{n-1} k a_k z^k \right|^2.$$

Integrating about $|z| = r$, $0 < r < 1$, gives

$$\sum_{k=2}^{n} k^2 |a_k|^2 r^{2k-2} + \sum_{k=n+1}^{\infty} |b_k|^2 r^{2k-2} \leq \alpha^2 \left\{ 4r^2 + \sum_{k=2}^{n-1} k^2 |a_k|^2 r^{2k} \right\}.$$

If we take the limit as r approaches 1, then

$$\sum_{k=2}^{n} k^2 |a_k|^2 \leq \alpha^2 \left\{ 4 + \sum_{k=2}^{n-1} k^2 |a_k|^2 \right\},$$

or

$$n^2 |a_n|^2 \leq 4\alpha^2 + \alpha^2 \sum_{k=2}^{n-1} k^2 |a_k|^2 - \sum_{k=2}^{n-1} k^2 |a_k|^2$$

$$= 4\alpha^2 + (\alpha^2 - 1) \sum_{k=2}^{n-1} k^2 |a_k|^2 \leq 4\alpha^2,$$

since $\alpha \leq 1$. Thus $|a_n| \leq 2\alpha/n$ for $n \geq 2$.

Sharpness of the inequality is shown by

$$f(z) = \int_0^z \frac{1 + \alpha t^{n-1}}{1 - \alpha t^{n-1}} \alpha t = z + \frac{2\alpha z^n}{n} + \frac{2\alpha^2}{2n-1} z^{2n-1} + \cdots.$$

THEOREM 2. *If $f(z)$ is in $D(\alpha)$, $0 < \alpha \leq 1$, then*
(i) *$f(z)$ maps $|z| < (\sqrt{2} - 1)/\alpha$ onto a convex domain if*

$$\left(\frac{(\sqrt{2} - 1)(\sqrt{3} + 1)}{\sqrt{2}} \right) \leq \alpha \leq 1,$$

(ii) *$f(z)$ maps*

$$|z| < [\{\alpha^2 - 1 + ((1 - \alpha^2)(1 + 4\alpha - \alpha^2))^{1/2}\}/2\alpha(1 + \alpha)]^{1/2}$$

onto a convex domain if

$$0 < \alpha \leq (\sqrt{2} - 1)(\sqrt{3} + 1)/\sqrt{2}.$$

The bounds in (i) *and* (ii) *are both sharp.*

PROOF.   Since $f(z)$ is in $D(\alpha)$, we have

$$f'(z) = (1 + \alpha z\theta(z))/(1 - \alpha z\theta(z)),$$

where $\theta(z)$ is analytic and $|\theta(z)| \leq 1$ for $z \in E$. Then

$$\frac{f''(z)}{f'(z)} = \frac{2\alpha\{z\theta'(z) + \theta(z)\}}{1 - \alpha^2 z^2[\theta(z)]^2}.$$

But for functions $\theta(z)$ [4, p. 168] we have

$$|\theta'(z)| \leq (1 - |\theta(z)|^2)/(1 - |z|^2).$$

Using this estimate we obtain

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{2\alpha|z|\,(|z| + |\theta(z)|)(1 - |z\theta(z)|)}{(1 - |z|^2)(1 - \alpha^2\,|z|^2\,|\theta(z)|^2)}.$$

Therefore, $|zf''(z)/f'(z)| \leq 1$ provided

(3) $2\alpha\,|z|\,(|z| + |\theta(z)|)(1 - |z\theta(z)|) \leq (1 - |z|^2)(1 - \alpha^2\,|z|^2\,|\theta(z)|^2).$

Letting $|z| = r$, $|\theta(z)| = x$ and $t = rx$, relation (3) becomes

(4) $2\alpha r(r + tr^{-1})(1 - t) \leq (1 - r^2)(1 - \alpha^2 t^2).$

We want to find the largest value of ρ such that (3) holds for all z such that $|z| < \rho$ and for all $\theta(z)$, $|\theta(z)| \leq 1$. This corresponds to finding the largest value of r for which (4) holds for all t, $0 \leq t \leq r$. Relation (4) becomes

(5) $H(t) \equiv (\alpha^2 r^2 - \alpha^2 + 2\alpha)t^2 + 2\alpha(r^2 - 1)t + (1 - 2\alpha r^2 - r^2) \geq 0.$

We want to determine the largest value of r for which $H(t) \geq 0$, $0 \leq t \leq r$. Then $f(z)$ will map $|z| < r$ onto a convex domain. Since $H'(t^*) = 0$ for $t^* = (1 - r^2)/[2 - \alpha(1 - r^2)]$ and $H''(t) > 0$, $H(t)$ assumes its minimum value at t^*. We separate the proof into two cases:

   Case A. $r < t^*$. Now $H(t)$ is nonincreasing on $[0, r]$, so $H(t) > H(r)$ for $0 \leq t \leq r$. Since $H(r) = (1 - r^2)(-\alpha^2 r^2 - 2\alpha r + 1)$, $H(r) \geq 0$, provided $\alpha^2 r^2 + 2\alpha r - 1 \leq 0$ or $r \leq (\sqrt{2} - 1)/\alpha$. Thus, $f(z)$ maps $|z| < (\sqrt{2} - 1)/\alpha$ onto a convex domain if $r = (\sqrt{2} - 1)/\alpha \leq t^*$. This restraint implies that α must lie in the interval $[\alpha_0, 1]$, where $\alpha_0 \equiv (\sqrt{2} - 1)(\sqrt{3} + 1)/\sqrt{2}$.

   The function $f(z) = -z - 2/\alpha \log(1 - \alpha z)$ shows this bound to be the best possible since

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 + 2\alpha z - \alpha^2 z^2}{1 - \alpha^2 z^2} = 0$$

for $z = (1 - \sqrt{2})/\alpha$.

Case B. $t^* \leq r$. The minimum value of $H(t)$ on $[0, r]$ occurs at t^*, so $H(t) \geq H(t^*)$. Therefore (5) will be satisfied if $H(t^*) \geq 0$. This inequality reduces to

$$(6) \qquad Q(r) \equiv \alpha(1 + \alpha)r^4 + (1 - \alpha^2)r^2 - (1 - \alpha) \leq 0.$$

But (6) is satisfied for $r < r_1$, where

$$r_1 \equiv [\{\alpha^2 - 1 + ((1 - \alpha^2)(1 + 4\alpha - \alpha^2))^{1/2}\}/2\alpha(1 + \alpha)]^{1/2}.$$

We shall show that if $0 < \alpha < \alpha_0$, then $t^* \leq r_1$. But $t^* \leq r$, if and only if $P(r) \equiv \alpha r^3 + r^2 + (2 - \alpha)r - 1 \geq 0$. Denote the zero of $P(r)$ in $(0, 1)$ by r_0. We shall show that $r_1 > r_0$ if $0 < \alpha < \alpha_0$. A tedious calculation shows that if $\alpha = \alpha_0$,

$$r_1 = \sqrt{2}/(1 + \sqrt{3}) = r_0 = (\sqrt{2} - 1)/\alpha_0.$$

Also for a fixed r and $\alpha < \alpha_0$, the expression $Q(r)$ increases with α. Thus, if (6) holds for a certain interval of values of r with $\alpha = \alpha_1$, then the condition holds for all $\alpha < \alpha_1$. Hence r_1 increases with decreasing α. But $\alpha = \alpha_0$ corresponds to $r_1 = r_0$. Thus for $\alpha < \alpha_0$, $r_1 > r_0$.

To show the estimate is sharp, we construct a function as follows. Let β be defined by

$$(7) \qquad r_1(r_1 - \beta)/(1 - \beta r_1) = (1 - r_1^2)/[2 - \alpha(1 - r_1^2)].$$

Since $r_1 > r_0$ for $\alpha < \alpha_0$, it follows that $(1 - r_1^2)/[2 - \alpha(1 - r_1^2)] < r_1$. Also, $r_1 < 1$. From (7) we have

$$0 < r_1(r_1 - \beta)/(1 - \beta r_1) < r_1,$$

or

$$(r_1 - \beta)/(1 - \beta r_1) < 1.$$

This implies that $(r_1^2 - 1)(1 - \beta^2) < 0$. But $r_1 < 1$, so $|\beta| < 1$. Define $\theta(z)$ by

$$(8) \qquad \theta(z) = (z - \beta)/(1 - \beta z).$$

Since $|\beta| < 1$, $|\theta(z)| \leq 1$ for $z \in E$. Define $f(z)$ by

$$f(z) = [1 - \alpha z \theta(z)]/[1 + \alpha z \theta(z)].$$

Then

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 - \alpha^2 z^2 [\theta(z)]^2 - 2\alpha z \theta(z) - 2\alpha z^2 \theta'(z)}{1 - \alpha^2 z^2 [\theta(z)]^2},$$

where

$$\theta'(z) = \{1 - [\theta(z)]^2\}/\{1 - z^2\}.$$

Using (7) and (8) and remembering that $Q(r_1) = 0$, we have that $1 + r_1 f''(r_1)/f'(r_1) = 0$. Therefore, $f(z)$ is not convex in $|z| < r$ if $r > r_1$.

THEOREM 3. *If $f(z)$ is in $D(\alpha)$ and if $L_r(f)$ denotes the length of the image of $|z|=r$ under $f(z)$, $0<r<1$, then $L_r(f)=O\{\log(1/(1-\alpha r))\}$, as $r\to1$.*

PROOF.

$$L_r(f) = \int_{|z|=r} |f'(z)|\,|dz| = \int_0^{2\pi} |f'(re^{i\theta})|\,r\,d\theta$$

$$\leqq \int_0^{2\pi} \left| \frac{1 + \alpha re^{i\theta}}{1 - \alpha r^{i\theta}} \right| r\,d\theta$$

$$= \int_0^{2\pi} \left| \frac{1 - \alpha^2 r^2 + 2\alpha r\,i\sin\theta}{1 - 2\alpha r\cos\theta + \alpha^2 r^2} \right| r\,d\theta$$

$$(9) \qquad \leqq r\int_0^{2\pi} \frac{1 - (\alpha r)^2}{1 - 2\alpha r\cos\theta + (\alpha r)^2}\,d\theta + \int_0^{2\pi} \frac{2\alpha r^2\,|\sin\theta|}{1 - 2\alpha r\cos\theta + \alpha^2 r^2}\,d\theta$$

$$= 2\pi r + 2r\int_0^{\pi} \frac{2\alpha r\sin\theta}{1 - 2\alpha r\cos\theta + \alpha^2 r^2}\,d\theta$$

$$= 2\pi r + 4r\log\frac{1 + \alpha r}{1 - \alpha r} = O\{\log 1/(1 - \alpha r)\}.$$

The first integral in (9) is a Poisson integral and the second can be evaluated directly.

REMARK. We state without proof that if $f(z) \in D(\alpha)$ and if $A_r(f)$ denotes the area of the image of $|z|<r$ under $f(z)$, $0<r<1$, then

$$A_r(f) \leqq \pi\{-3 - (4/\alpha^2 r^2)\log(1 - \alpha^2 r^2)\}.$$

The inequality is sharp.

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