ASYMPTOTIC INVERSION OF LAPLACE TRANSFORMS:
A CLASS OF COUNTEREXAMPLES

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Abstract. Let \( f \) be a complex-valued locally integrable function on \([0, +\infty)\), and let \( Lf \) be its Laplace transform, whenever and wherever it exists. We review some known methods, exact and approximate, for recovering \( f \) from \( Lf \). Since numerical algorithms need auxiliary information about \( f \) near \(+\infty\), we note that the behavior of \( f \) near \(+\infty\) depends on the behavior of \( Lf \) near \(0+\), hence that our ability to retrieve \( f \) is limited by the class of momentless functions, namely, all functions \( f \) such that \( Lf(s) \) converges absolutely for \( \text{Re}(s)>0 \) and satisfies

\[
Lf(s) = o(s^n) \quad \text{near } 0+ \quad \text{for } n = 0, 1, 2, \ldots.
\]

We discuss the space \( Z \) of momentless functions and complex distributions, then construct a family of elements in this space which defy various plausible conjectures.

1. Introduction. Let \( f \) be a complex-valued locally integrable function on \([0, +\infty)\), and let

\[
L[f; s] = \int_0^\infty \exp(-st)f(t)\, dt
\]

be its Laplace transform, whenever and wherever this integral exists. Indeed [21, pp. 96–102] for some \( \sigma_a(f) \) and \( \sigma_c(f) \) with \(-\infty \leq \sigma_c(f) \leq \sigma_a(f) \leq +\infty\) there are maximal half-planes \( \text{Re}(s) > \sigma_a(f) \) and \( \text{Re}(s) > \sigma_c(f) \) in which respectively \( L[f; s] \) is absolutely convergent and conditionally convergent. If these half-planes are nonvoid, then \( L[f; s] \) is also holomorphic at least in \( \text{Re}(s) > \sigma_c(f) \), and \( f(t) \) is uniquely determined by \( L[f; s] \) except on a set of measure zero [21, pp. 99, 108].

An important step in many problems is the inversion of a Laplace transform, that is, the recovery of \( f(t) \) from \( L[f; s] \). Sometimes this can be accomplished exactly through transform tables (e.g. [5]), inversion formulas ([3, p. 286], [6], [21, pp. 108, 141]), or convergent series ([3, pp. 301–305], [17, p. 97], [18], [19, Chapter 9]). However many inversions...
employ numerical techniques ([1], [2]), which typically lose accuracy for large $t$; hence such algorithms require further information which describes $f(t)$ approximately near $+\infty$. Theorems which derive the limiting behavior of $f(t)$ from that of $L[f; s]$ are called respectively Tauberian or inverse Abelian according as they involve extra hypotheses on $f(t)$ or on $L[f; s]$.

If $f(t)$ can be expressed by the inversion integral [21, p. 108]

$$f(t) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \exp(ts)L[f; s] \, ds,$$

if $L[f; s]$ can be continued analytically to the left, and if the contour of (1.2) can be moved sufficiently in that direction, then the behavior of $f(t)$ near $+\infty$ is determined by that of $L[f; s]$ near its rightmost singularities. If these singular points are all poles then their contributions are simple residues and [4, p. 110]

$$\sum_{m=0}^{\infty} P_m(t) \exp[a(m)t] \text{ near } +\infty,$$

with $P_m$ a polynomial for each $m$, and $\text{Re}[a(m)] \downarrow -\infty$ as $m \to \infty$. An essential singularity of $L[f; s]$ yields a Taylor expansion for $P_m$ [3, p. 488].

If $L[f; s]$ has a branch point $s_0$ as its unique rightmost singularity, then $s_0$ may be shifted to the origin without loss of generality. Thus for any positive $a$ we can infer $\int_0^t f(u) \, du \sim ct^a / \Gamma(1+a)$ near $+\infty$ by the Tauberian theorem of Karamata [19, p. 197], given that $L[f; s] \sim cs^{-a}$ near 0+ and that $f(t) + k t^{a-1} \geq 0$ for some $k$. Also for any complex $a$ we can infer that $f(t) \sim ct^{a-1} / \Gamma(a)$ near $+\infty$ by inverse Abelian theorems of Doetsch, given that $L[f; s] \sim cs^{-a}$ near 0 in a sector $|\arg s| \leq \theta$, where either $\theta > \pi/2$ or $\theta = \pi/2$, and in the latter case $L[f; s]$ satisfies further conditions on the imaginary axis. For generalized functions some results of this kind have been proved by Lavoine [12], involving the regularized functions $ct^{a-1}$; for $\log f(t)$ some estimates near $+\infty$ have been obtained by Wagner ([17], [18], [25]), describing still more singular behavior.

To get sharper results at a branch point, we consider series

$$f(t) \sim \sum_{m=0}^{\infty} P_m(t) \log t^{a(m)} \text{ near } +\infty$$

with $P_m$ and $\text{Re}[a(m)]$ as in (1.3). Expansions near $+\infty$ of this form, or near 0+ with $\text{Re}[a(m)] \uparrow +\infty$, were originally treated, it seems, by Mellin [13], and are thus called Mellin series or expansions by the author. If the function $f$ has a Mellin series near $+\infty$ then this series for $f$, and certain values of its Mellin transform, determine systematically a Mellin series.
for $L[f; s]$ near $0+$ [7]; while this series for $L[f; s]$ and the existence of an expansion (1.4) determine uniquely the Mellin series for $f(t)$ near $+\infty$ ([8], [9]). The existence and form of a series (1.4) follows by two inverse Abelian theorems of Doetsch ([4, pp. 150–160], [9], [11]) from assumptions on $L[f; s]$ in a sector $|\arg s| \leq \theta$, where either $\theta > \pi/2$ or $\theta = \pi/2$, and in the latter case $L[f; s]$ satisfies further conditions on the imaginary axis. However these assumptions on $L[f, s]$ are not necessary [9, Example 4]. These results of Doetsch have also been extended in work of Riekstina ([23], [24]).

To explore all possibilities for theorems of this kind, we remark that if $g$ is rapidly decreasing near $+\infty$ then $L[g; s]$ can be expanded by moments:

$$L[g; s] \sim \sum_{n=0}^{\infty} \mu_n (-s)^n / n! \quad \text{near} \quad 0+ \quad \text{with} \quad \mu_n = \int_0^{\infty} t^n g(t) \, dt.$$  

Thus any transformable $g$ will be called a momentless function if $\sigma_n (g) \leq 0$ and

$$L[g; s] = o(s^n) \quad \text{near} \quad 0+ \quad \text{for all} \quad n = 0, 1, 2, \cdots.$$  

A nontrivial example from standard tables [5, p. 158] is

$$g(t) = t^{-1/2} \cos(kt)^{1/2} \quad \text{with} \quad k > 0,$$

$$L[g; s] = (\pi/s)^{1/2} \exp(-k/4s).$$

If $g$ is a function of this kind then $L[f; s]$ and $L[f+g; s]$ have identical Mellin series near $0+$. Thus the Mellin series for $f$ is not recoverable unless all permissible $g$ are rapidly decreasing under the set of hypotheses for a conjectured theorem. We shall therefore construct a class of momentless functions and distributions through which we may eliminate a number of conjectures on asymptotic inversion.

2. Notation. We shall construct the desired counterexamples on $[0, +\infty)$ as a family of functions and measures, but can introduce the associated concepts more easily in a space of generalized functions. Indeed if $D'_+$ is the space of Schwartz distributions on $(-\infty, +\infty)$ with support in $[0, +\infty)$ then $D'_+$ is a commutative algebra over the complex field under “pointwise” addition, scalar multiplication, and the standard convolution ([15, pp. 113, 121], [22, pp. 122–130]). This convolution $f \ast g$ for elements of $D'_+$ extends the definition

$$[f \ast g](t) = \int_0^t f(t - u) g(u) \, du$$

for functions on $[0, +\infty)$ [15, p. 115].
Within $D_+^\prime$, let $e$ represent the Dirac delta "function", so that $e$ is the identity for this algebra, and let $1_+$ denote the Heaviside step function, so that

$$[1_+ \ast f](t) = \int_0^t f(u) \, du$$

for functions on $[0, +\infty)$. Then we can define

$$f \ast 0 = e, \quad f \ast 1 = f, \quad f \ast n+1 = f \ast f \ast n$$

for any $f$ in $D_+^\prime$ and all $n=1, 2, \cdots$. Moreover $D_+^\prime$ is closed under

$$f \mapsto 1_+ \ast f, \quad f \mapsto df/dt,$$

and the first of these mappings is the inverse of the second.

For any element $f$ of $D_+^\prime$ and any complex $s=c+iu$ the Laplace transform $L[f; s]$ is defined ([15, p. 217], [22, p. 222]) as the Fourier transform

$$L[f; s] = \int_0^\infty \exp(-itu) \exp(-ct) f(t) \, dt$$

whenever $\exp(-ct)f(t)$ is in the space $S'$, so that (2.5) is a well-defined entity. Then for some value $\sigma(f)$, either real or $\pm \infty$, the transform $L[f; s]$ is defined and analytic on $\text{Re}(s) > \sigma(f)$, and for functions on $[0, +\infty)$ this half plane of existence includes the preceding $\text{Re}(s) > \sigma_0(f)$ ([15, p. 218], [22, p. 223]).

Now consider the set $A$ of all $f$ in $D_+^\prime$ such that $L[f; s]$ is defined in this sense for $\text{Re}(s) > 0$ at least and such that

$$L[f; s] = O(s^k) \quad \text{for some real } k$$

as $s \to 0$ in this half plane. Clearly $A$ is a subalgebra of $D_+^\prime$ by the convolution theorem ([15, p. 222], [22, p. 240]), and is closed under the mappings (2.4) by the identities ([15, pp. 222–223], [22, p. 228])

$$L[1_+ \ast f; s] = s^{-1}L[f; s], \quad L[df/dt; s] = sL[f; s].$$

Call $f$ momentless if it lies in $A$ and satisfies (1.6); define $Z$ as the set of all such $f$. Then $Z$ is an ideal in $A$ by (1.6) and (2.6), while $Z$ is closed under (2.4) by (1.6) and (2.7).

Within $A$ denote by $J$ the space of all elements $f$ which correspond to locally integrable functions, modulo the space of all functions which vanish except on null sets. The introduction of $J$ offers a criterion for $A$: if $f$ is given in $D_+^\prime$ then $f$ is also in $A$ whenever $1_+^n \ast f$ is in $J$ for some $n=0, 1, 2, \cdots$, and

$$[1_+^n \ast f](t) = o(t^k) \quad \text{near } +\infty$$
for some $k > 0$. Indeed, under these conditions $L[1^*n \ast f; s]$ is absolutely convergent on $\text{Re}(s) > 0$ and is $o(s^{-k+1})$ as $s \to 0$ ([19, p. 182], [22, p. 249]); so that $L[f; s]$ satisfies (2.6) by use of (2.7). However this criterion is not necessary, for

$$
(2.9) \quad g(t) = \sum_{n=0}^{\infty} (d/dt)^n e(t - n)
$$

is in $A$, but no $1^*n \ast g$ is in $J$.

Let $M$ be the set of all $f$ in $A$ which correspond to complex measures on $[0, +\infty)$, namely, those for which $1^* \ast f$ has locally bounded variation. Then $M$ is a subalgebra of $A$ [19, p. 84]; measure algebras are discussed in standard works ([10, pp. 141–150], [14, pp. 13–17]). Let $C^n$ be the set of all $f$ in $J$ which have $n$ continuous derivatives on $(-\infty, +\infty)$, for all $n=0, 1, 2, \cdots$ or $\infty$. Then $J$ is an ideal in $M$ [10, p. 143], all $C^n$ are ideals in $M$ [15, p. 122], and $C^\infty$ is closed under (2.4).

Finally, we collect these remarks on algebraic structure to obtain the following ideals in the system $M$:

$$
(2.10) \quad Z \cap M, Z \cap J, \quad Z \cap C^n \quad \text{for } n=0, 1, \cdots, \infty.
$$

Therefore we can generate elements of $Z \cap M$ with arbitrary preassigned smoothness from a special family $\{h_{a,x}\}$ with $a$ and $x$ suitable real numbers. Also we can construct more singular elements of $Z$ by repeated differentiation of $h_{a,x}$.

3. Construction. For any real $a$ and $x$ the expression

$$
(3.1) \quad K(a, x, z) = (1-z)^{-1-a} \exp[xz/(z-1)]
$$

is analytic in the complex $z$ plane cut from 1 to $+\infty$, and is the generating function [16, equation (5.1.9)] for the generalized Laguerre polynomials $L_n^{(a)}(x)$, so that

$$
(3.2) \quad K(a, x, z) = \sum_{n=0}^{\infty} L_n^{(a)}(x) z^n \quad \text{for } |z| < 1.
$$

Moreover, by Fejer's formula [16, equation (8.22.1)],

$$
L_n^{(a)}(x) = \pi^{-1} \exp(x/2)x^{(-2a-1)/4} \cdot n^{(2a-1)/4} \cdot \cos[2(nx)^{1/2} - (2a + 1)\pi/4] + O[n^{(2a-3)/4}] \quad \text{as } n \to +\infty
$$

uniformly on any compact interval in $0 < x < +\infty$.

On $|z| \leq 1$ the series (3.2) converges absolutely for $a < -\frac{3}{2}$ by the last formula, and conditionally for $a = -\frac{3}{2}$ by Littlewood's theorem [21, p. 215]. However for each positive $x$ the set

$$
(3.4) \quad \{2(nx)^{1/2} - (2a + 1)\pi/4: n = 0, 1, 2, \cdots \} \quad \text{modulo } 2\pi
$$
is dense in $[0, 2\pi)$, so that, with any positive $\delta$ and $n_0$,

$$
|L_n^{(a)}(x)| \leq (1 - \delta)n^{-1} \exp(x/2) \cdot x^{(2a-1)/4} \cdot n^{(2a-1)/4}
$$

for some $n > n_0$. Thus $L_n^{(a)}(x)$ for large $n$ is not $O(n^r)$ unless $r \geq (2a-1)/4$.

For any fixed real $a$ and positive $x$, letting $e(t)$ be the Dirac delta function, we construct

$$
h_{a,x}(t) = \sum_{n=0}^{\infty} L_n^{(a)}(x)e(t - n).
$$

By Abel's theorem [21, pp. 27-28] if $a \leq -\frac{3}{2}$ then

$$
[1_+ \ast h_{a,x}](+\infty) = \sum_{n=0}^{\infty} L_n^{(a)}(x) = \lim_{z \to 1-} K(a, x, z) = 0.
$$

By this relation and (3.3), if $a$ is arbitrary then

$$
|1_+ \ast h_{a,x}(t)| = O[(t^{(2a+3)/4}]
$$
as $t \to +\infty$.

Hence by (2.8) these $h_{a,x}$ are elements of $M$ with support on the nonnegative integers. Moreover these $h_{a,x}$ are elements of $Z$, since if $\text{Re}(s) > 0$ then

$$
L[h_{a,x}; s] = \sum_{n=0}^{\infty} L_n^{(a)}(x)\exp(-ns) = K(a, x, \exp(-s)) = o(s^m)
$$

near $0^+$ for $m = 0, 1, 2, \cdots$.

**Example 1.** Before finding this construction the author advanced the conjecture that if $f$ were an element of $Z$ which was bounded as a measure on $[0, +\infty)$ then

$$
|1_+ \ast f|(+\infty) - |1_+ \ast f|(0) = o(t^{-n})\quad\text{near } +\infty\quad\text{for all } n > 0.
$$

However if $a < -\frac{3}{2}$ then $h_{a,x}$ is a bounded measure on $[0, +\infty)$ by (3.3), and $h_{a,x}$ is algebraically decaying near $+\infty$ by (3.5). Indeed some multiple of $h_{a,x}$, by (3.7), is the difference of two probability measures on the integers; hence these measures differ by a momentless distribution which decays no faster than $t^{(2a-1)/4}$.

**Example 2.** One might suppose that matters would improve for functions $f$ in $Z \cap L^1(-\infty, +\infty)$. However let $f = g \ast h_{a,x}$, where $g$ is intuitively any function in $L^1[0, +\infty)$, or technically any function in $J \cap L^1(-\infty, +\infty)$. Then by (2.10), $f$ is in $Z \cap J$ for all real $a$, and by (3.3), $f$ is in $Z \cap L^1(-\infty, +\infty)$ for $a < -\frac{3}{2}$. Moreover if $g$ is unbounded at the origin and is bounded outside each neighborhood of zero, then $f$ or some multiple is unbounded at $t = 0, 1, 2, \cdots$ and is the difference between two probability densities.

**Example 3.** One might expect more from the intersection of $Z$ with some Sobolev space. However if $g$ is $C^\infty$ with support in $(0, 1)$ and if
\( f = g \ast h_{a,z} \) with \( h_{a,z} \) as defined, then \( f \) is in \( Z \cap C^\infty \) by (2.10) and \( f \) is in \( L^p(-\infty, +\infty) \) by (3.3) whenever

\[
\sum_{n=1}^{\infty} n^{(2a-1)p/4} < +\infty,
\]

hence for all \( p \) in \( [1, +\infty) \) whenever \( a < -\frac{1}{2} \). Moreover \( f^{(m)} = g^{(m)} \ast h_{a,z} \), so that \( f^{(m)} \) has the same properties as \( f \), and thus \( f^{(m)} \) is in \( L^p(-\infty, +\infty) \) for all \( p \) in \( [1, +\infty) \) and all \( m = 0, 1, 2, \cdots \). At the same time \( f \) and its derivatives decay no faster than \( t^{(2a-1)/4} \).

These examples show that conditions on \( f \) of smoothness and integrability cannot produce estimates of \( f(t) \) near \(+\infty\) from estimates of \( L[f;s] \) near \( 0+ \). Indeed for any fixed \( a \) and \( x \), the numbers

\[
\{ L_n^{(a)}(x) : n = 0, 1, 2, \cdots \}
\]

change sign according to (3.3), so that the functions \( f = g \ast h_{a,z} \) oscillate systematically as \( t \to +\infty \). Thus the positivity condition of Karamata’s theorem serves to exclude some elements of \( Z \). However this theorem applies only to functions \( f \) for which \( 1+ \ast f \) grows algebraically near \( +\infty \), whence new results might well arise from other hypotheses under which \( f \) oscillates negligibly near \( +\infty \).

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