LATTICE SUMS OF HOMOGENEOUS FUNCTIONS

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Abstract. An asymptotic estimate for a general lattice sum is derived using Fourier analysis techniques.

1. Let \( x_1, \ldots, x_n \) be a basis of \( \mathbb{R}^n \), \( n \geq 2 \), and \( L = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n \) the lattice they generate. Let \( B(T) = \{ x \mid \|x\| \leq T \} \) and \( L(T) = \sum_{L \cap B(T)} 1 \). Then it is known that \( L(T) = (c_n/d)T^n + O(T^{n(n-1)/(n+1)}) \) where \( c_n \) = Volume of \( B(1) \) and \( d = |\text{det}(x_1, \ldots, x_n)| \). We wish to prove an analogous estimate for a general homogeneous function.

2. Let \( f(x) \) be a real-valued homogeneous function on \( \mathbb{R}^n \), i.e., \( f(rx) = r^\alpha f(x) \) for all \( r \geq 0 \) with the restriction that \( \alpha \geq 0 \). We then define \( c_f = \int_{B(1)} f(x) \, dx \) and \( L_f(T) = \sum_{x \in L \cap B(T)} f(x) \). We will prove

Theorem. If \( f \) is homogeneous and \( f \in C^{[n/2]+1}(\mathbb{R}^n) \) then \( L_f(T) = (c_f/d)T^{n+\alpha} + O(T^{n(n-1)/(n+1) + \alpha}) \).

3. The Theorem may be obtained by the methods used in [1]. The only thing to prove is that the Fourier transform of \( f \cdot X_{B(1)}(x) \) is \( O(\|x\|^{-(n+1)/2}) \) where \( X_{B(1)}(x) \) is the characteristic function of \( B(1) \):

\[
\hat{f} \cdot X_{B(1)}(x) = \int_{B(1)} e^{2\pi i (y, x)} f(y) \, dy = \frac{1}{2\pi i \|x\|} \int_{\partial B(1)} e^{2\pi i (y, x)} g_1(y, x) \, dy
\]

with \( g_1(y, x) \) \([n/2]\)-times differentiable in \( y \) (see the Lemma in [1] and Lemma 3 in [3]). By applying a partition of unity to \( \partial B(1) \) and integrating by parts as in [2] our estimate follows. The proof of the Theorem is now carried out exactly as in [1] using the Poisson summation formula.

Bibliography


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Received by the editors October 13, 1972. 
Key words and phrases. Lattice point, homogeneous function.

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