A NOTE ON THE RADON-NIKODYM THEOREM

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Abstract. This paper gives a necessary and sufficient condition in order that a bounded linear mapping from $L^1(\mu)$ into a Banach space be compact. It is applied to provide a slightly improved form of the Radon-Nikodym theorem for vector valued measures and to give a sufficient condition in order that the range of a vector valued measure of bounded variation be compact.

In this paper, we give a sufficient condition in order that a measure is representable as an indefinite Bochner integral. We will show that a bounded linear operator $T$ mapping $L^1(\mu)$ into a Banach space is compact if the set $\{T(\psi_{M_i}/\mu(M_i)) : M_i \in \Sigma\}$ is precompact for every sequence of disjoint measurable sets $\{M_i\}$ (see Theorem 1). Combining this with a result of Rieffel [2] gives a slightly improved Radon-Nikodym theorem for the Bochner integral.

We apply this result in Theorem 2 to give a sufficient condition in order that the range of a vector valued measure be precompact. Here, we follow the approach of Uhl [3].

Lemma. Let $(X, \Sigma, \mu)$ be a finite measure space. Let $T:L^1(\mu) \to F$ be a bounded linear operator, where $F$ is a Banach space. For each positive real number $c$, define $\mathcal{R}(c) = \{T(\psi_{M_j}/\mu(M_j)) : M_j \in \Sigma, 0 < \mu(M_j) < c\}$ where $\psi_{M_j}$ denotes the characteristic function of $M_j$. Then, $\mathcal{R}(b)$ is a precompact set if and only if there is a number $a$ such that $0 < a < b$ and such that $\mathcal{R}(a)$ is precompact.

Proof. Suppose that there is an $a$ with $0 < a < b$ such that $\mathcal{R}(a)$ is precompact. Since the measure space is finite there are at most a finite number of atoms $A_1, A_2, \ldots, A_k$ in $\Sigma$ with $a < \mu(A_i), i = 1, 2, \ldots, k$. Let $Q(a) = \{T(\psi_{A_i}/\mu(A_i)) : i = 1, 2, \ldots, k\}$ then $\mathcal{R}(a) \cup Q(a)$ is precompact. Suppose that $y \in \mathcal{R}(b)$, i.e. $y = T(\psi_M/\mu(M))$ for some $M \in \Sigma$ with $0 < \mu(M) < b$. By Saks' theorem (Dunford-Schwartz [1, Lemma IV 9-7]) there exists a finite sequence of disjoint measurable sets $M_1, M_2, \ldots, M_n$ such that $M_i$...
is either an atom with $a < \mu(M_i)$ or $0 < \mu(M_i) < a$ and $M = \bigcup_{i=1}^{n} M_i$. Hence,

$$y = T(\psi_{M_i}/\mu(M)) = \sum_{i=1}^{n} \frac{\mu(M_i)}{\mu(M)} T\left(\frac{\psi_{M_i}}{\mu(M_i)}\right)$$

is a member of the convex hull of the precompact set $R(a) \cup Q(a)$. Therefore, the closure of $R(b)$ is a subset of the compact set $\text{cl}(\text{co}(R(a) \cup Q(a)))$, the closed convex hull of the set $R(a) \cup Q(a)$, and $R(b)$ is precompact.

**Theorem 1.** Let $(X, \Sigma, \mu)$ be a finite positive measure space and let $F$ be a Banach space. Then a bounded linear operator $T: L^1(\mu) \to F$ is compact if the set $\{T(\psi_{M_i}/\mu(M_i)) : M_i \in \Sigma\}$ is precompact for every sequence of disjoint measurable sets $\{M_i\}$.

**Proof.** We may assume that $\mu(X) = 1$. It suffices to show that $R(1)$ is precompact since the image of the positive functions of the unit ball of $L^1(\mu)$ is the closed convex hull of $R(1)$. Indeed, if $\|f\|_1 = 1$ and $f > 0$ then for a given $\epsilon > 0$ there is a simple function $\sum_{i=1}^{n} c_i \psi_{M_i}$, $c_i > 0$, such that

$$\left\| \sum_{i=1}^{n} c_i \psi_{M_i}\right\|_1 = 1 \quad \text{and} \quad \left\| f - \sum_{i=1}^{n} c_i \psi_{M_i}\right\|_1 < \epsilon.$$

Now $\sum_{i=1}^{n} c_i \psi_{M_i} = \sum_{i=1}^{n} c_i \mu(M_i)(\psi_{M_i}/\mu(M_i))$ is a member of the convex hull of $R(1)$ since $\sum_{i=1}^{n} |c_i| |\mu(M_i)| = \| \sum_{i=1}^{n} c_i \psi_{M_i} \|_1 = 1$. By the preceding Lemma, it suffices to show that there exists an $a$ with $0 < a < 1$ such that $R(a)$ is precompact. Suppose the contrary, i.e., none of $R(a)$ is precompact for each $a$ with $0 < a < 1$; then there is an $\epsilon > 0$ such that none of $R(a)$ can be covered by a finite number of $\epsilon$-balls $B(y_i, \epsilon) = \{y \in F : \|y - y_i\| < \epsilon\}$. Let $y_1 \in R(1)$ and by induction choose $y_n \in R(1/n) = \bigcup_{i=1}^{n} B(y_i, \epsilon)$. The sequence $\{y_n\}$ is clearly infinite and has no convergent subsequence. Since $y_n \in R(1/n)$ there is a measurable set $M_n$ such that

$$y_n = T(\psi_{M_n}/\mu(M_n)), \quad n = 1, 2, \ldots,$$

and $\mu(M_n) \leq 1/n$. Choose a subsequence $\{\alpha_i\}$ of $\{\mu(M_n)\}$ such that

(1) \quad $\alpha_{i+1}/\alpha_i < 1/2^i, \quad i = 1, 2, \ldots$.

Now

(2) \quad $\sum_{j \geq i} \alpha_j < \alpha_{i+1} + \frac{1}{2^i+1} \alpha_{i+1} + \frac{1}{2^{i+3}} \alpha_{i+1} + \cdots < \frac{3}{2} \alpha_{i+1}$.

Let $\alpha = \mu(M_{n(i)})$ and let $N_i = M_{n(i)} = \bigcup_{j > i} M_{n(j)}$. Clearly $N_i$ and $N_j$ are
disjoint if \( i \neq j \) and
\[
\mu(N_i) \geq \mu(M_{n(i)}) - \sum_{j \neq i} \mu(M_{n(j)})
\]
\[
\geq \alpha_i - \frac{3}{2} \alpha_{i+1} \geq \alpha_i - \frac{3}{2} \cdot \frac{1}{2^i} \alpha_i \quad \text{by (2) and (1)}
\]
\[
= \alpha_i \left(1 - \frac{3}{2^{i+1}}\right) > 0, \quad i = 1, 2, 3, \ldots,
\]
and
\[
\frac{\mu(N_i)}{\mu(M_{n(i)})} = \frac{\mu(N_i)}{\alpha_i} \geq 1 - \frac{3}{2^{i+1}}, \quad i = 1, 2, 3, \ldots.
\]
Now
\[
\left\| T\left(\frac{\psi_{N_i}}{\mu(N_i)}\right) - T\left(\frac{\psi_{M_{n(i)}}}{\mu(M_{n(i)})}\right)\right\| \leq \left\| T\right\| \left\| \frac{\psi_{N_i}}{\mu(N_i)} - \frac{\psi_{M_{n(i)}}}{\mu(M_{n(i)})}\right\|
\]
\[
\leq \left\| T\right\| \left\{ \frac{\mu(N_i)}{\mu(M_{n(i)})} - \frac{1}{\mu(M_{n(i)})} + \frac{\mu(\bigcup_{j \neq i} M_{n(j)})}{\mu(M_{n(i)})} \right\}
\]
\[
\leq \left\| T\right\| \left\{ \frac{3}{2^{i+1}} + \frac{3 \alpha_{i+1}}{2^i \alpha_i} \right\} \quad \text{by (2) and (3)}.
\]
\[
= \frac{3}{2^i} \left\| T\right\| \to 0 \quad \text{as} \quad i \to \infty.
\]
Therefore we conclude that the sequence \( \{T(\psi_{N_i}/\mu(N_i))\} \) has no convergent subsequence, which contradicts the hypothesis of the theorem.

Remark. If the measure space \((X, \Sigma, \mu)\) is atom-free, the space need not be finite in order that the theorem hold.

Corollary. Let \((X, \Sigma, \mu)\) be a \(\sigma\)-finite positive measure space. Let \(\phi: \Sigma \to F\) be a vector valued measure of bounded variation, where \(F\) is a Banach space. Then \(\phi\) is the indefinite integral with respect to \(\mu\) of a Bochner integrable function \(f: X \to F\) if and only if

(1) \(\phi(M) = 0\) whenever \(\mu(M) = 0\), where \(M \in \Sigma\);

(2) \(\phi\) has a finite total variation;

(3) given any \(M \in \Sigma\) with \(0 < \mu(M) < \infty\), there is a set \(N \in \Sigma\) so that \(N \subseteq M\), \(\mu(N) > 0\) and \(N\) satisfies the following condition: if \(\{N_i\}\) is any sequence of disjoint (nonnull) measurable sets in \(N\), then \(\{\phi(N_i)/\mu(N_i)\}\) is a precompact set.

For the sufficiency, we apply Rieffel's theorem in [2] and note that the condition (3) given here implies the condition (3) of Rieffel, by using Theorem 1 above. (It is not difficult to check that by setting \(T_N(S) = \phi(S)\),...
for each measurable set $S \subseteq N$, we obtain a bounded linear operator $T_N : L^1(N, \Sigma_N, \mu) \to F$ where $\Sigma_N$ is the family of measurable sets contained in $N$. Indeed, $\|T_N(S)\| \leq |\phi|(S)$ where $|\phi|$ is the total variation of $\phi$. By the standard Radon-Nikodym Theorem, $|\phi|$ has a Radon-Nikodym derivative with respect to $\mu$. Denoting this derivative by $g$, we get $\|T_N(S)\| \leq \int_S |g| \, d\mu$.

If we assume that $g$ is bounded over $N$, as we may by replacing $N$ with a suitable subset, if necessary, then $\|T_N(S)\| \leq C \int_S d\mu$, where $C$ is that bound.) The necessity of the condition is implied by Rieffel’s theorem.

Let $\phi : \Sigma \to F$ be a vector valued measure of bounded variation and let $\mu$ be the total variation of $\phi$, i.e., for each $M \in \Sigma$, $\mu(M) = \sup \sum_i \|\phi(M_i)\|$ where the sup is taken over all possible sequences of disjoint subsets $M_i$ of $M$. It is shown in Uhl [3] that a sufficient condition in order that the range of $\phi$ be precompact is that the Banach space $F$ be either a reflexive space or a separable dual space. In Uhl’s proof the condition that the Banach space is either reflexive or separable dual is utilized to ensure that the measure $\phi$ admits a Bochner kernel representation, i.e., there is a Bochner integrable function $f$ in $L^1(X, \Sigma, \mu)$ such that $\phi(M) = \int_M f \, d\mu$. We will apply the preceding theorem, using this approach of Uhl, to give a sufficient condition in order that the range of $\phi$ be precompact.

**Theorem 2.** If there exists a sequence $\{X^i\}$ of measurable subsets of $X$ such that $\mu(X^i) < \infty$ for each $i$, $\mu(X - \bigcup_{i=1}^\infty X^i) = 0$ and such that for each $i$ the set $\{\phi(M_i^k) | \mu(M_i^k) ; M_i^k \subseteq X^i \}$ is precompact for every sequence $\{M_i^k\}$ of disjoint measurable subsets of $X^i$, then the range of $\phi$ is precompact.

**Proof.** Let the operator $T : L^1(X, \Sigma, \mu) \to F$ be a linear extension of $\phi$ such that $T(\alpha \psi_M + \beta \psi_N) = \alpha \phi(M) + \beta \phi(N)$ for characteristic functions $\psi_M$, $M \in \Sigma$. Then the hypothesis of the theorem ensures that the restriction of the operator $T$ to $L^1(X^i, \Sigma, \mu)$ is compact for each $i$ (i.e., $T$ is “locally compact”). Without loss of generality, we may assume that $\{X^i\}$ is a disjoint sequence of measurable sets. Therefore, by an inductive application of the Dunford-Pettis-Phillips theorem, there exists a Bochner integrable function $f : X \to F$ such that $T(g) = \int g f \, d\mu$ for each $g \in L^1(X, \Sigma, \mu)$. Now select a sequence $\{\psi_n\}$ of simple functions with their values in $F$ such that $\lim_n \int |f - \psi_n| \, d\mu = 0$. Define $T : L^1(X, \Sigma, \mu) \to F$ by $T_n(g) = \int g \psi_n \, d\mu$ for each $n$. Then, by Hölder’s inequality $T_n$ is bounded for each $n$ and $\lim_n \|T_n - T\| \leq \lim_n \|f - \psi_n\| \, d\mu = 0$. Since $\psi_n$ is simple, $T_n$ is finite dimensional and hence is compact, thus, $T$ also is compact. Since $T$ is a linear extension of $\phi$, the range of $\phi$ is precompact.

**Remark.** This theorem extends Uhl’s results. For, if the Banach space $F$ is either a reflexive space or a separable dual space, then, by Dunford-Pettis-Phillips theorem, every bounded linear operator $T : L^1(X, \Sigma, \mu) \to F$
has a Bochner kernel; using Egorov's theorem it is seen that $T$ is "locally compact" so that the hypothesis of the above theorem is satisfied. At the same time, the hypothesis of the theorem can be satisfied even when $F$ is neither reflexive nor a separable dual, as we see in the following example.

**Example.** Let $X = [0, \infty)$, $X_n = [n, n+1]$ and $\Sigma$ be the $\sigma$-field of Borel sets of $X$ and let $\lambda$ denote the Lebesgue measure on $X$. Define a vector valued measure $\phi: \Sigma \to c_0$ by

$$\phi(M) = (\lambda(M \cap X_1), \frac{1}{2}\lambda(M \cap X_2), \ldots, (1/2^{n-1})\lambda(M \cap X_{n-1}), \ldots),$$

$M \in \Sigma$.

Clearly $\phi$ is a measure with values in $c_0$ and $\|\phi\| \leq 1$. The restriction of $\phi$ to $X_n$ is $\phi|_{X_n}(M) = (0, 0, \ldots, (1/2^{n-1})(M \cap X_{n-1}), 0, \ldots)$ and it is compact. For any subset $M$ of $X_n$, clearly $\phi(M)\mu(M) = (0, 0, \ldots, 1, 0, \ldots)$ and $\{\phi(M)\mu(M): M \subseteq X_n\}$ is compact.

**References**


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