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# MODEL COMPANIONS FOR $\aleph_0$-CATEGORICAL THEORIES

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ABSTRACT. We show that any countable $\aleph_0$-categorical theory without finite models has a model companion (which is also $\aleph_0$-categorical). We find the model companions for all $\aleph_0$-categorical abelian groups, and conclude with some remarks on the $\aleph_1$-categorical case.

1. **Introduction.** The main objective of this paper is to prove that any countable $\aleph_0$-categorical theory $K$ without finite models has a model companion (which is also $\aleph_0$-categorical).

We recall that $K'$ is a model companion of $K$ if $K$ and $K'$ are theories in the same first-order language, $K'$ is model complete, and $K$ and $K'$ are mutually model consistent, i.e. any model of $K$ can be embedded in a model of $K'$ and vice versa. (Model completeness is discussed in [**5**]; model companions are introduced in [**1**].) This concept is a generalization of that of model completion [**5**]; $K'$ is a model completion of $K$ if $K'$ extends $K$, any model of $K$ can be embedded in a model of $K'$, and for any model $A$ of $K$ and models $B_1$, $B_2$ of $K'$ such that $A \subset B_1$ and $A \subset B_2$, we have $\langle B_1, a\rangle_{a\in A} \equiv \langle B_2, a\rangle_{a\in A}$, i.e. $B_1$ and $B_2$ are elementarily equivalent in a language which has constants for all the elements of $A$. A model completion of $K$ is, of course, a model companion of $K$, but a theory may have a model companion without having a model completion. We remark that a model companion, when it exists, is unique up to equivalence: any two model companions of $K$ have the same models (see, for example, [1, Theorem 5.3]).

If to the assumptions on $K$ indicated in the first paragraph we add the requirement that $K$ be inductive, i.e. that the union of any chain of models of $K$ be itself a model of $K$, then $K$ is itself already model complete, by a result of Lindstrom [**4**, p. 189]. However if $K$ is not assumed inductive then under the given hypotheses $K$ need not even have a model completion. Consider, for example, the case where $K_0$ is the familiar set of axioms for

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the concept of a densely ordered set with (say) last element but no first element. Then $K_0$ is $\aleph_0$-categorical by the familiar back-and-forth argument of Cantor, and the theory $K_0'$ of densely ordered sets without first or last elements is a model companion for $K$. But since $K_0'$ does not extend $K_0$, the uniqueness result mentioned above implies that $K_0$ has no model completion.

In relinquishing inductivity we give up a very useful model theoretic property; it is nice to know that we still get a model companion, even if not a model completion.

2. **Preliminaries.** The general framework in which we shall work is that of infinite forcing in model theory; we assume familarity with [**6**]. Let $\Sigma$ denote the class of all substructures of models of $K$, equivalently the class of all models of $K_\forall$, the set of universal consequences of $K$. Robinson defines a relation "$A$ forces $\theta$" between elements $A$ of $\Sigma$ and sentences $\theta$ defined in $A$, as follows, by induction on the structure of $\theta$:

(i) If $\theta$ is atomic, $A$ forces $\theta$ iff $A \models \theta$.

(ii) If $\theta$ is $\varphi \wedge \psi$, $A$ forces $\theta$ iff $A$ forces $\varphi$ and $A$ forces $\psi$.

(iii) If $\theta$ is $\varphi \vee \psi$, $A$ forces $\theta$ iff $A$ forces $\varphi$ or $A$ forces $\psi$ (or both).

(iv) $A$ forces $\exists y \theta(y)$ iff $A$ forces $\theta(c)$ for some constant $c$ denoting an element of $A$.

(v) $A$ forces $\neg\theta$ iff for no $B$ in $\Sigma$ which extends $A$ is it the case that $B$ forces $\theta$.

We say that $A$ weakly forces $\theta$ iff no extension of $A$ in $\Sigma$ forces $\neg\theta$. If $\theta$ is defined in $A$, this is equivalent to saying that $A$ forces $\neg\neg\theta$.

A structure $A$ in $\Sigma$ is said to be generic if for any $\theta$ defined in $A$, $A \models \theta$ iff $A$ forces $\theta$. Using the fact that $\Sigma$ is inductive it can be shown [**6**, Theorem 2.4] that any $A \in \Sigma$ has an extension $B \in \Sigma$ which is generic; since it is immediate by properties of forcing that if $A$ and $B$ are generic and $A \subset B$ then $A \prec B$, we see that if $K'$ is a set of sentences having precisely the generic structures as its models, then $K'$ is a model companion for $K$.

The following special case of a theorem of Robinson [**6**] will enable us to find such a $K'$ for our $\aleph_0$-categorical $K$.

THEOREM. *Let $\theta(x_1, \cdots, x_n)$ be a formula of the form $\neg\varphi(x_1, \cdots, x_n)$ which is defined in K. There exists a set $S_\theta$ of sets $\mu = \{\theta_\tau^\mu(x_1, \cdots, x_n) \mid \tau$ in some index set $I_\mu\}$ of existential formulas defined in K such that for any $a_1, \cdots, a_n$ denoting elements of some $A \in \Sigma$,*

$$A \text{ forces } \theta(a_1, \cdots, a_n) \leftrightarrow A \models \bigvee_{\mu \in S_\theta} \bigwedge_{\tau \in I_\mu} \theta_\tau^\mu(a_1, \cdots, a_n).$$

We will denote $\bigvee_{\mu \in S_\theta} \bigwedge_{\tau \in I_\mu} \theta_\tau^\mu(x_1, \cdots, x_n)$ by $\theta'(x_1, \cdots, x_n)$.

A result of Carol Coven says that $A$ is generic iff for any $\theta$ defined in $A$, $A \models \theta$ iff $A$ weakly forces $\theta$. (For a proof see [6, Theorem 6.4].) From this and the theorem just quoted it follows that $A$ is generic iff it is a model of

$$(*) \qquad K_\forall \cup \{\forall x_1 \cdots \forall x_n(\theta(x_1, \cdots, x_n) \leftrightarrow (\neg\neg\theta)'(x_1, \cdots, x_n))\},$$

where $\theta$ varies over all formulas defined in $K$.

REMARK. We have used Coven's characterization of genericity instead of the definition—i.e. weak forcing instead of forcing—in order to be able to apply the special case of Robinson's theorem. If we were to use forcing, we would have to resort to the general form of Robinson's theorem, for $\theta$ not necessarily a negation; but then the resulting $\theta'$'s would have additional quantifiers in them, which would cause difficulties with our proof.

To replace the infinitary axiomatization $(*)$ by a first-order one we will (not surprisingly) use the following result on $\aleph_0$-categorical theories.

THEOREM (RYLL-NARDJEWSKI). *Let $K$ be a complete countable theory without finite models. Then the following are equivalent*:

(i) *$K$ is $\aleph_0$-categorical.*

(ii) *For each $n$, $K$ has only finitely many $n$-types.*

(iii) *For each $n$, every $n$-type in $K$ is principal.*

For the relevant definitions and a proof of the theorem see [9, p. 91].

Using this we will show that within $K$ $(\neg\neg\theta)'(x_1, \cdots, x_n)$ is equivalent to a first-order formula. Then we will use the relationship between $K$ and its forcing companion $K^F$ (the set of sentences in the language of $K$ which hold in all generic structures) to transfer this equivalence to $K^F$, and this will yield the finitary axiomatization $K'$.

We will define $K'$ to include $K^F$, but it will follow from the definition of $K^F$ and the fact that $K'$ axiomatizes the class of generic structures that in fact $K' = K^F$. This is in accordance with Theorem 5.4 of [6], which implies that the model companion of $K$, if it exists, must be logically equivalent to $K^F$.

## 3. **The main results.**

THEOREM 1. *Let $K$ be a countable $\aleph_0$-categorical theory with no finite models. $K$ has a model companion.*

PROOF. As indicated above, we will show that the infinitary axiomatization

$$K_\forall \cup \{\forall x_1 \cdots \forall x_n(\theta(x_1, \cdots x_n) \leftrightarrow (\neg\neg\theta)'(x_1, \cdots, x_n))\}$$

of the class of generic structures can be replaced by a finitary one.

First we claim that for any $\theta(x_1, \cdots, x_n)$ there exists a universal predicate $\theta^*(x_1, \cdots, x_n)$ defined in $K$ such that for any model $A$ of $K$

$$A \models \forall x_1 \cdots \forall x_n(\neg(\neg\neg\theta)'(x_1, \cdots, x_n) \leftrightarrow \theta^*(x_1, \cdots, x_n)).$$

To see this, first consider any disjunct $\bigwedge_r (\neg\neg\theta)^\mu_r(x_1, \cdots, x_n)$ of $(\neg\neg\theta)'$. Define a formula $\varphi_\mu(x_1, \cdots, x_n)$ as follows. If $\{(\neg\neg\theta)^\mu_r(x_1, \cdots, x_n)\}$ is not consistent with $K$, let $\varphi_\mu$ be $x_1 \neq x_1$. If on the other hand $\{(\neg\neg\theta)^\mu_r(x_1, \cdots, x_n)\}$ is consistent with $K$, then it can be extended to a complete $n$-type in $K$. By Ryll-Nardjewski's theorem we can let $\{p_1, \cdots, p_r\}$ be the set of all those $n$-types in $K$ which extend $\{(\neg\neg\theta)^\mu_r(x_1, \cdots, x_n)\}$, and we can let $\varphi_i$, $1 \leqq i \leqq r$, be a generator of $p_i$. Define $\varphi_\mu = \bigvee_{i=1}^r \varphi_i$. In either case, for any model $A$ of $K$,

$$A \models \forall x_1 \cdots \forall x_n\left(\bigwedge_r (\neg\neg\theta)^\mu_r(x_1, \cdots, x_n) \leftrightarrow \varphi_\mu(x_1, \cdots, x_n)\right).$$

By the same argument we see that there is a formula $\psi(x_1, \cdots, x_n)$ such that if $A \models K$ then

$$A \models \forall x_1 \cdots \forall x_n\left(\bigwedge_\mu \neg\varphi_\mu(x_1, \cdots, x_n) \leftrightarrow \psi(x_1, \cdots, x_n)\right);$$

therefore

$$A \models \forall x_1 \cdots \forall x_n(\neg(\neg\neg\theta)'(x_1, \cdots, x_n) \leftrightarrow \psi(x_1, \cdots, x_n)).$$

This implies that if $A$ and $B$ are models of $K$, $A \subset B$, $a_1, \cdots, a_n \in A$, and $B \models \psi(a_1, \cdots, a_n)$, then $B$ does not weakly force $\theta(a_1, \cdots, a_n)$, whence $A$ does not weakly force it either, so $A \models \psi(a_1, \cdots, a_n)$. Therefore $\psi$ is persistent under restriction with respect to $K$, so by a fundamental result of A. Robinson [5] there exists a universal predicate $\theta^*(x_1, \cdots, x_n)$ such that

$$K \vdash \forall x_1 \cdots \forall x_n(\psi(x_1, \cdots, x_n) \leftrightarrow \theta^*(x_1, \cdots, x_n)).$$

Clearly $\theta^*$ satisfies our claim.

Next we claim that the equivalence between $\neg(\neg\neg\theta)'$ and $\theta^*$ transfers from $K$ to $K^F$. To see this we observe that if $A$ is any model of $K^F$ then there exists a model $B$ of $K$ with $A \subset B$ such that $A$ is existentially complete in $B$, i.e. any existential sentence with constants from $A$ which holds in $B$ holds already in $A$. The existence of such a $B$ follows from the fact, established in [6], that every universal-existential consequence of $K$ is a consequence of $K^F$, either by a standard ultraproduct construction or by direct appeal to Theorem 3.4.3 of [5]. Now if $a_1, \cdots, a_n \in A$ then by the above

$$B \models \neg(\neg\neg\theta)'(a_1, \cdots, a_n) \quad \text{iff} \quad B \models \theta^*(a_1, \cdots, a_n).$$

Since $\theta^*$ is universal and $A$ is existentially complete in $B$, the right side of this equivalence is equivalent to $A \models \theta^*(a_1, \cdots, a_n)$. Similarly since $\neg(\neg\neg\theta)'(a_1, \cdots, a_n)$ is equivalent to a conjunction of disjunctions of universal sentences, the left side is equivalent to $A \models \neg(\neg\neg\theta)'(a_1, \cdots, a_n)$. Therefore

$$(**) \qquad A \models \forall x_1 \cdots \forall x_n(\neg(\neg\neg\theta)'(x_1, \cdots, x_n) \leftrightarrow \theta^*(x_1, \cdots, x_n)).$$

Consider now the set of axioms

$$K' = K^F \cup \{\forall x_1 \cdots \forall x_n(\neg\theta(x_1, \cdots, x_n) \leftrightarrow \theta^*(x_1, \cdots, x_n))\}$$

where $\theta$ varies over predicates defined in $K$. Any generic structure is a model of $K^F$; therefore by (**) it is for any $\theta$ a model of

$$\forall x_1 \cdots \forall x_n(\neg(\neg\neg\theta)'(x_1, \cdots, x_n) \leftrightarrow \theta^*(x_1, \cdots, x_n)),$$

and since it is generic it is a model of

$$\forall x_1 \cdots \forall x_n(\neg(\neg\neg\theta)'(x_1, \cdots, x_n) \leftrightarrow \neg\theta(x_1, \cdots, x_n)).$$

Therefore it is a model of $K'$.

Conversely, a model of $K'$ is clearly a model of $K_\forall$ since $K_\forall \subset K^F$; since it is a model of $K'$, it satisfies

$$\forall x_1 \cdots \forall x_n(\neg\theta(x_1, \cdots, x_n) \leftrightarrow \theta^*(x_1, \cdots, x_n))$$

for any $\theta$ by definition, and it satisfies

$$\forall x_1 \cdots \forall x_n(\neg(\neg\neg\theta)'(x_1, \cdots, x_n) \leftrightarrow \theta^*(x_1, \cdots, x_n))$$

by (**). Therefore it satisfies

$$\forall x_1 \cdots \forall x_n(\neg\theta(x_1, \cdots, x_n) \leftrightarrow \neg(\neg\neg\theta)'(x_1, \cdots, x_n)),$$

so it is generic by the infinitary axiomatization.

This completes the proof; we reiterate that $K' = K^F$, so that $K^F$ is model complete.

COROLLARY. *If $K_1$ is mutually model consistent with a theory $K_2$ which is $\aleph_0$-categorical and has no finite models (and is in the same countable language as $K_1$), then $K_1$ has a model companion.*

PROOF. If $K_1$ and $K_2$ are mutually model consistent then $K_1^F = K^F$ (see [**6**, Theorem 3.10]), and $K_2^F$ is model complete by the theorem.

We observe that in the situation of Theorem 1, the model companion is also $\aleph_0$-categorical:

THEOREM 2. *The model companion $K'$ of a countable $\aleph_0$-categorical theory $K$ without finite models is $\aleph_0$-categorical.*

PROOF. Since $K$ is complete by the Los-Vaught theorem, it has the joint embedding property, so $K'$ is complete [**6**, Theorem 4.4]. Also since for any $n$ the sentence

$$\exists x_1 \cdots \exists x_n \left( \bigwedge_{1 \leqq i,j \leqq n; i \neq j} x_i \neq x_j \right)$$

is a theorem of $K$, it is a theorem of $K'$, so $K'$ has no finite models. Therefore by the Ryll-Nardjewski theorem it suffices to show that for each $n$ there are only finitely many $n$-types in $K'$. Since $K'$ is model complete every $n$-type in $K'$ is determined by its existential formulas. Given any such type $S$ form $S'$ by adding to the set of existential formulas in $S$, for each existential $\varphi$ not in $S$, a universal formula logically equivalent to $\neg\varphi$. Any $S'$ is realized in a model $A$ of $K'$ and hence (as in the proof of Theorem 1) in a model $B$ of $K$ such that $A \subset B$ and $A$ is existentially complete in $B$. Hence $S'$ can be extended to a complete type in $K$. Now if $S_1 \neq S_2$ then $S_1'$ contains a formula logically equivalent to the negation of a formula in $S_2'$, so $S_1'$ and $S_2'$ extend to different types in $K$. Since there are only finitely many $n$-types in $K$, this implies that there are only finitely many $n$-types $S$ in $K'$.

4. **An example.** As an illustration of Theorem 1, we find the model companions for all $\aleph_0$-categorical abelian groups.

A group $G$ is said to be $\aleph_0$-categorical if Th$(G)$ is $\aleph_0$-cateogrical. In [7] Rosenstein proves that an abelian group is $\aleph_0$-categorical iff it is of bounded order, i.e. there is a bound on the orders of its elements. Our existence theorem then says that for any abelian group $G$ of bounded order, Th$(G)$ has a model companion. (If $G$ is finite Th$(G)$ is clearly model complete, so we assume that $G$ is infinite, so that Th$(G)$ has no finite models.) It is easy to see that the model companion of Th$(G)$ is a theory of abelian groups of bounded order and is $\aleph_0$-categorical; in fact in this case the $\aleph_0$-categoricity is clear without Theorem 2, because the model companion is complete and is therefore the theory of any of its models, which is $\aleph_0$-categorical by Rosenstein's result. So we are claiming that to any such $G$ we can associate a countable abelian group m.c.$(G)$ of bounded order, unique up to isomorphism, such that the model companion of Th$(G)$ is Th(m.c.$(G)$). Given $G$ we now find m.c.$(G)$ explicitly.

$G$ has the form

$$\bigoplus_{i=1}^{s} [Z^{r(p_i^{k_i})}(p_i^{k_i}) \oplus Z^{r(p_i^{k_i+n_{i,1}})}(p_i^{k_i+n_{i,1}}) \oplus \cdots \oplus Z^{r(p_i^{k_i+n_{i,m_i}})}(p_i^{k_i+n_{i,m_i}})]$$

for primes $p_1, \cdots, p_s$, where $k_i$ and $n_{i,u}$, $1 \leqq u \leqq m_i$, are natural numbers, $0 < n_{i,1} < \cdots < n_{i,u}$, and $r(p_i^{k_i+n_{i,u}})$, $1 \leqq u \leqq m_i$, is a cardinal number. (Of course $Z^r(n)$ denotes the direct sum of $r$ copies of $Z(n)$, the cyclic group with $n$ elements. See [**2**, Theorem 6].)

If $H$ is any such group and $\mathrm{Th}(H)$ is model complete then for any $i$, $r(p_i^{k_i+n_{i,u}})$ must be finite for $1 \leqq u \leqq m_i$. For if $r(p_i^{k_i+n_{i,u}})$ is infinite consider an embedding of $H$ into itself which maps the generator of a $Z(p_i^{k_i})$ term onto the generator of a $Z(p_i^{k_i+n_{i,u}})$ term times $p_i^{n_{i,u}}$. Such an embedding is clearly not elementary.

Now given $G$ it is not hard to see that the unique countable model of $\mathrm{Th}(G)$ is the group $G_0$ obtained from $G$ by replacing each infinite $r(p_i^{k_i+n_{i,u}})$ by $\aleph_0$, and that $\mathrm{m.c.}(G)=\mathrm{m.c.}(G_0)$. Thus to find $\mathrm{m.c.}(G)$ we must find some countable group $H$ which is at least of the special type indicated in the last paragraph, and such (by model consistency and $\aleph_0$-categoricity) that $G_0$ and $H$ are embeddable in each other. It is not hard to show that the only possible such group—again we know there is one by Theorem 1— is that obtained by performing the following operation on $G_0$.

THEOREM 3. *For any $\aleph_0$-categorical abelian group $G$, the model companion of* $\mathrm{Th}(G)$ *is* $\mathrm{Th}(\mathrm{m.c.}(G))$, *where* $\mathrm{m.c.}(G)$ *is obtained from $G_0$ by, in the part corresponding to any $p_i$, deleting everything to the left of the part corresponding to the largest $u$ for which $r(p_i^{k_i+n_{i,u}})$ is infinite.* (*If there is no such $u$ we do nothing.*)

As a check we can show directly that $\mathrm{Th}(H)$ is in fact model complete for $H$ of the special form described above, by showing that $\mathrm{Th}(H)$ is inductive and using the result of Lindstrom referred to in §1.

Another illustration of Theorem 1 is obtained by observing that for any infinite $\aleph_0$-categorical linear ordering $S$, $\mathrm{Th}(S)$ has as its model companion the theory of dense linear order without endpoints. The required model consistency can be established by using the characterization of $\aleph_0$-categorical linear orderings given by Rosenstein in [**8**].

5. **Some remarks on the $\aleph_1$-categorical case.** If in Theorem 1 we replace $\aleph_0$-categoricity by $\aleph_1$-categoricity, then we can draw the conclusion that the class of generic structures can be axiomatized in $L_{\omega_1\omega}$, the extension of first-order logic obtained by allowing the formation of countable conjunctions and disjunctions. (For information on $L_{\omega_1\omega}$ see [**3**].) For we know that

$$K^F \cup \left\{\forall x_1 \cdots \forall x_n \left(\theta(x_1, \cdots, x_n) \leftrightarrow \bigvee_{\mu \in S_{\neg\neg\theta}} \bigwedge_{\tau \in I_\mu} (\neg\neg\theta)^\mu_\tau(x_1, \cdots, x_n)\right)\right\}$$

is an axiomatization. If $K^F \vdash \forall x_1 \cdots \forall x_n(\neg\theta(x_1, \cdots, x_n))$ we may replace the right side by $x_1 \neq x_1$; otherwise we can assume that for any disjunct $\bigwedge_{\tau \in I_\mu} (\neg\neg\theta)^\mu_\tau(x_1, \cdots, x_n)$ the set $\{(\neg\neg\theta)^\mu_\tau(x_1, \cdots, x_n)\}$ is precisely the set of all existential formulas which hold at $a_1, \cdots, a_n$ in some model $A$ of $K^F$ which weakly forces $\theta(a_1, \cdots, a_n)$. Then by using the device of extending to a model $B$ of $K$ such that $A$ is existentially complete in $B$, as in the proof of Theorem 2, we can conclude that if there were uncountably many different disjuncts there would be uncountably many $n$-types in $K$, contradicting its $\aleph_1$-categoricity **[10]**.

We conclude with the obvious remark that the full strength of $\aleph_1$-categoricity is not needed here; all we need is that there are only countably $n$-types in $K$ for each $n$.

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