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A NOTE ON MORDELL'S EQUATION $y^2 = x^3 + k$

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This paper is dedicated to my distinguished teacher,
Professor Basil Gordon, University of California at Los Angeles.

ABSTRACT. In this note we prove that $\limsup_{k\to\infty} N'(k) \geqq 6$, where $N'(k)$ is the number of integral solutions of $y^2=x^3+k$ with $(x, y)=1$.

Let $N(k)$ be the number of solutions (x, y) in integers of $y^2=x^3+k$. Since the curve $y^2=x^3+7$ has no rational points **[1]**, the same is true of the curves $y^2=x^3+7t^6$, where t is any integer. Then $N(7t^6)=0$. Hence $\liminf_{k\to\infty} N(k)=0$.

On the other hand, $\limsup_{k\to\infty} N(k)=\infty$, as the following argument shows. The equation $y^2=x^3+3$ has the integral solution $(1, 2)$, and hence it has infinitely many rational solutions, by **[2]**. Choose n of these solutions (x_i, y_i) $(1\leqq i\leqq n)$. Write the fractions x_i, y_i with a common denominator i.e. $x_i=a_i/d$, $y_i=b_i/d$, where a_i, b_i and $d\in Z$. Then

$$(b_i/d)^2 = (a_i/d)^3 + 3, \quad \text{or} \quad (b_i d^2)^2 = (a_i d)^3 + 3d^6.$$

Thus $N(3d^6)\geqq n$. From this the result follows. We note that $(x, y)\neq 1$ here.

Now let $N'(k)$ be the number of integral solutions of $y^2=x^3+k$ with $(x, y)=1$. It appears likely that $\limsup_{k\to\infty} N'(k)\neq\infty$, for $N'(k)$, corresponding to points of finite order on an elliptic curve seems to be uniformly bounded **[3]**. Thus if $\limsup_{k\to\infty} N'(k)=\infty$ is to be true, it must be shown that there are elliptic curves whose Mordell-Weil group has very large rank and that each generator of infinite order should have rational integral coordinates. This has not been proved yet. However we can at least prove the following result.

THEOREM 1. *$\limsup_{k\to\infty} N'(k)\geqq 6$.*

PROOF. Consider the identities
(i) $(\pm(t^3-3))^2=2^3+(t^6-6t^3+1)$,
(ii) $(\pm(t^3+1))^2=(2t)^3+(t^6-6t^3+1)$,
(iii) $(\pm(3t^3-1))^2=(2t^2)^3+(t^6-6t^3+1)$.

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It is easily seen that if t is even, then

$$(t^3 - 3, 2) = (t^3 + 1, 2t) = (3t^3 - 1, 2t^2) = 1.$$

Hence if k is any integer of the form $t^6 - 6t^3 + 1$, and t even, we have $N'(k) \geqq 6$ with $(x, y) = 1$, which completes the proof.

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