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A COMBINATORIAL ANALOG OF LYAPUNOV'S THEOREM FOR INFINITESIMALLY GENERATED ATOMIC VECTOR MEASURES

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ABSTRACT. It is shown that the range of a measure obtained by the addition of infinitesimal vectors is convex up to infinitesimal errors.

By a well-known theorem of Lyapunov, the range in R^n of a nonatomic totally finite vector-measure is convex (see [**5**]). Recent developments in economics (see [**3**]) have established the need for a similar theorem for a purely atomic measure on a *finite set A, where the measure of each atom $a \in A$ is a vector of infinitesimal length. The desired result is a corollary of the following theorem of Steinitz ([**7**, pp. 167–172]; also see [**1**, pp. 148–149]): Given any finite collection of vectors in n-space R^n with sum 0 and maximum norm M, there is an ordering $v_1, v_2, \cdots, v_p$ of those vectors such that the norm of any partial sum $\sum_{i=1}^{j} v_i$ is smaller than $2nM$. (Note that the norm in Steinitz's theorem need not be the Euclidean norm, and if it is, $2nM$ may not be the best constant for dimension n.) (See [**2**] and [**4**].)

Let $*R$ and $*N$ denote the nonstandard models for the real numbers R and natural numbers N in a fixed enlargement of a structure that contains R. (See [**6**].) We write $a \simeq b$ when $a \in *R$, $b \in *R$ and $a-b$ is infinitesimal, and we denote by $|v|$ the Euclidean distance of any vector v from the origin 0. Fix $n \in *N$, n finite or infinite. If u and v are vectors in $*R^n$, we shall write $u \simeq v$ if $|u-v| \simeq 0$. Recall that a *finite set is a set for which there is an internal one-to-one correspondence with an initial segment of $*N$; such a set has all the "formal" properties of a finite set.

THEOREM. *Let A be a *finite set, and for each $a \in A$, let $v(a)$ be a vector in n-space $*R^n$ with $nv(a) \simeq 0$. For each internal set $B \subset A$, set*

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$S(B)=\sum_{a\in B} v(a)$. *Then the following is true*:

(i) *If* $S(A)\not\simeq 0$, *there is an ordering* $v_1, v_2, \cdots, v_p$ *of the set* $\{v(a): a \in A\}$ *so that, for each* $j \in {}^*N$ *with* $1 \leqq j \leqq p$, $\sum_{i=1}^{j} v_i \simeq \lambda S(A)$ *where* $\lambda \in {}^*R$.

(ii) *Given internal sets* $B \subset A$ *and* $C \subset A$ *and given* $\lambda \in {}^*R$ *with* $0<\lambda<1$, *there is an internal set* $D \subset A$ *with* $S(D) \simeq \lambda S(B) + (1-\lambda)S(C)$.

PROOF. To prove (i), let P_L denote the projection of ${}^*R^n$ onto the line L through 0 and $S(A)$, and let P_H denote the projection of ${}^*R^n$ onto the hyperplane H perpendicular to L at 0. The sum of the vectors $\{P_H(v(a)): a \in A\}$ is 0. Therefore by Steinitz's theorem, there is an ordering $v_1, v_2, \cdots, v_p$ of the set $\{v(a): a \in A\}$ so that, for any $j \in {}^*N$ with $1 \leqq j \leqq p$,

$$\left| P_H \sum_{i=1}^{j} v_i \right| = \left| \sum_{i=1}^{j} P_H(v_i) \right| \leqq 2n \max_{a \in A} |v(a)| \simeq 0,$$

whence $\sum_{i=1}^{j} v_i \simeq P_L \sum_{i=1}^{j} v_i = \lambda S(A)$ for some $\lambda \in {}^*R$. This proves (i) and more, for given $\lambda \in {}^*R$ with $0<\lambda<1$, there is a first $\omega \in {}^*N$ such that $|\sum_{i=1}^{\omega} P_L(v_i)| \geqq \lambda |S(A)|$, and $\sum_{i=1}^{\omega} P_L(v_i)$ and $S(A)$ are on the same side of H. There is thus an $\omega \in {}^*N$ for which $\sum_{i=1}^{\omega} P_L(v_i) \simeq \lambda S(A)$. We now obtain statement (ii) from this fact by replacing A with $B-C$ and $C-B$ and noting that, for $0<\lambda<1$,

$$\lambda S(B) + (1-\lambda)S(C) = \lambda S(B-C) + (1-\lambda)S(C-B) + S(B \cap C).$$

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