A CASE IN WHICH IRREDUCIBILITY OF AN ANALYTIC
GERM IMPLIES IRREDUCIBILITY
OF THE TANGENT CONE

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Abstract. There are simple examples in which a variety is irreducible at a point but has a reducible tangent cone. The following theorem is proved. If \( X_p \) is an irreducible analytic germ and if the Jacobian ideal becomes principal on the normalization then the tangent cone of \( X \) at \( p \) is irreducible. If, moreover, the singular set of \( X \) is a manifold at \( p \) then \( X \) is Whitney \( a,b \)-regular along the singular set at \( p \).

Let \( X \) be a pure \( r \)-dimensional analytic subset of open \( U \) in \( \mathbb{C}^n \). \( \mathcal{O}^n \) denotes the holomorphic structure sheaf on \( U \), \( I \) denotes the (self-radical) ideal sheaf defining \( X \) in \( U \), and \( \mathcal{O} \) denotes the resulting (reduced) holomorphic structure on \( X \). Let \( T=(T_1, \ldots, T_n) \) be coordinates on \( \mathbb{C}^n \). Hereafter \( J \) denotes the Jacobian ideal of \( X \) on \( X \). It is a coherent sheaf of ideals whose stalk at \( p \) is obtained as follows. Take the \( \infty \times n \) matrix whose entries are of the form \( (\partial f/\partial T_j)_{X, j=1, \ldots, n, f \in I_p} \). \( J_p \) is the ideal in \( \mathcal{O}_p \) generated by the \( (n-r) \times (n-r) \) subdeterminants of this matrix. It is sufficient to restrict \( f \) to a system of generators for \( I_p \) in which case one obtains a system of generators for \( J_p \). Thus we can assume that \( f \) belongs to a finite set \( \mathcal{F} \) of generators for \( I_p \). By replacing \( U \) by a smaller open set we may also assume \( f \) is holomorphic on \( U \), \( \forall f \in \mathcal{F} \), \( \{f_q, f \in \mathcal{F}\} \) generates \( I_q \), and the \( (n-r) \times (n-r) \) subdeterminants obtained from the matrix

\[
(\partial f/\partial T_j)_{X, j=1, \ldots, n, f \in \mathcal{F}}
\]

have germs at \( q \) which generate the ideal \( J_q \) for all \( q \) in \( X \). Let \( t=\#(\mathcal{F}) \).

The singular set of \( X \), hereafter denoted \( Sg(X) \), is the locus of \( J \). Let \( \pi: (X', \mathcal{O}') \rightarrow (X, \mathcal{O}) \) denote the normalization of \( X \). The following are

Received by the editors July 17, 1972.


Key words and phrases. Tangent cone, Jacobian ideal, normalization, Whitney \( a,b \)-regularity.

\(^1\) This work was done while the author was a visiting staff member at Purdue University. The manuscript was prepared while the author was supported by NSF Grant GP-20139 Amendment Number 2 at the University of Notre Dame. The manuscript was revised while the author was a staff member at George Mason University.

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in natural one-to-one correspondence: \( \{ p'_1, \ldots, p'_k \} = \pi^{-1}(p) \), \( \{ p_1, \ldots, p_k \} = \) minimal primes of \( \mathcal{O}_p \), \( \{ X_1, \ldots, X_k \} = \) irreducible components of the germ of \( X \) at \( p \). We say \( J^{O'} \) is locally principal over \( p \) if and only if \( J_p^{O^{O'}} \) is principal for \( \lambda = 1, \ldots, k \). This is an open condition on \( X \).

Let \( T(X) = (p, u) \in X \times C^n | \sum f_i(p)u_i = 0 \ \forall f \in \mathcal{F} \) where \( f_i = \partial f/\partial T_j \). \( T(X, p) \) denotes the fiber of the projection of \( T(X) \) to \( X \). The singular points of \( X \) are those for which \( \dim T(X, p) > r \). We let \( Y \) denote the set of singular points. \( C_t(X) \) is the closure in \( T(X) \) of \( T(X - Y) \). \( C_t(X, p) \) is the fiber of \( C_t(X) \) over \( p \). It is the union of a collection of \( r \)-dimensional subspaces of \( C^n \). Let \( G^{n-1,r-1} \) be the Grassmann variety of \( r \)-dimensional subspaces of \( C^n \). If \( q \) is simple on \( X \), \( T(X, q) \) has a unique point \( \tau(X, q) \) in \( G^{n-1,r-1} \). Let \( \tau(X) \) denote the closure of \( \{(q, \tau(X, q)) | q \in X - Y \} \) in \( X \times G^{n-1,r-1} \). Let \( \tau(X, p) \) denote the fiber of the projection to \( X \). See [6, Theorem 5.1, p. 218 and Theorem 7.1, p. 224]. There is a natural map \( \varphi: C_t(X) \rightarrow \tau(X) \) which commutes with the projections to \( X \) and makes \( C_t(X) \) a fiber space of constant fiber dimension \( n - r \) over \( \tau(X) \). \( \tau(X) \rightarrow X \) is a proper modification of \( X \) which is an isomorphism exactly over \( X - Y \). Denote the structure sheaf on \( \tau(X) \) by \( \mathcal{O} \).

Let \( (B_J(X), \mathcal{O}) \) denote the blowing-up of \( X \) along \( J \). This is a subspace of \( X \times P^n \) which can be realized as follows. Let \( \mathcal{F} \) denote the functions on \( X \) obtained from (*)). These are determinants which we enumerate in the following way for our subsequent convenience. Let \( \mathcal{F} = (f_1, \ldots, f_i) \) and let \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \). By \( J_{\lambda} \) we mean the tuple obtained from (*) by taking subdeterminants from the \((n-r) \times n \) submatrix of (*) whose rows involve the functions \( f_{ij}, \ j = 1, \ldots, n-r \). Let \( J_{\lambda}, k = 1, \ldots, (n-r) \) denote the entry of this tuple ordered in some fixed manner. Let \( \lambda = (\cdots, J_{\lambda}, \cdots) \) be a fixed ordering of the tuples \( J_{\lambda} \). Introduce indeterminants \( Z_{\lambda}^k \) ordered in the same manner. Then \( (B_J(X), \mathcal{O}) \) is the analytic space defined by the equations \( \{ Z_{\lambda}^k f_j^\mu - Z_{\lambda}^\mu f_j^k = 0, \ \forall \lambda, \mu, k, j \text{ subject to the above conventions} \} \) in \( X \times P^{N-1} \) where the \( Z_{\lambda}^k \) are homogeneous coordinates on \( P^{N-1} \) and \( N = (n-1)(n-r) \). \( B_J(X) \rightarrow X \) blows up exactly over \( X - Y \). We denote the fiber of this projection by \( B_J(X, p) \).

The tangent cone to \( X \) at \( p \), denoted \( C(X, p) \), is \( \{ u \in C^n | f^*(u) = 0 \ \forall f \in I_p \} \) where \( f^* \) denotes the leading form at \( p \) of \( f \). Since \( G(\mathcal{O}_p) \), the associated graded ring of \( \mathcal{O}_p \) with respect to its maximal ideal, is isomorphic to \( C[T]/A \) where \( A \) is the ideal generated by \( \{ f^*(T) | f \in I_p \} \), there is a one-to-one correspondence between the components of the cone \( C(X, p) \) and the minimal primes of \( G(\mathcal{O}_p) \). We use this and the fact, [6, (3.1), p. 212], that \( C_t(X, p) = C(X, p) \) to prove the theorem of this paper.

**Proposition 1.** With the notation and assumptions of the preceding paragraphs, the following are equivalent:
(i) \( \dim C_4(X, p) = r \),
(ii) \( \tau(X, p) \) is finite,
(iii) \( B_J(X, p) \) is finite,
(iv) \( J \) is locally principal over \( p \) on \( (X', \mathcal{O}') \),
(v) \( C_4(X, p) = C(X, p) \).

Moreover, if these conditions hold then \( \#\tau(X, p) \leq \#B_J(X, p) \leq \#\{\text{irreducible components of } X_p\} \) and \( C(X, p) \) is a union of \( \#\tau(X, p) \) \( r \)-dimensional subspaces of \( C^n \).

**Proof.** As remarked earlier \( C_4(X, p) \) is the union of \( r \)-dimensional subspaces of \( C^n \). \( \tau(X, p) \) is the union of the Grassmann coordinates of these subspaces. Both \( C_4(X, p) \) and \( \tau(X, p) \) are algebraic sets. Clearly \( \dim C_4(X, p) = r \) if and only if \( \tau(X, p) \) is finite which proves the equivalence of (i) and (ii).

Indices have the meanings established in the description of \( B_J(X) \). We will show that to each point of \( \tau(X, p) \) there corresponds at least one and at most finitely many points of \( B_J(X, p) \). (This is always true. Moreover, when \( X \) is a complete intersection \( (\tau(X), \mathcal{O}) \) and \( (B_J(X), \mathcal{O}_{\text{red}}) \) are isomorphic. In [4, Remark (1.2), p. 3] it seems to be stated that they are always isomorphic. This does not appear to be true to us although we have been unable to produce a counterexample.) Let \( (p, \alpha) \in \tau(X) \). Let \( a \) be affine coordinates for \( \alpha \). There exist simple points \( p_v \rightarrow p, c_v \in C, \) such that \( c_v J_\mu(p_v) \rightarrow a \) for at least one \( \mu \). Moreover, given any \( \mu \) such that \( J_\mu(p_v) \neq 0 \) for almost all \( v \) a sequence \( d_v \) exists such that \( d_v J_\mu(p_v) \rightarrow a \) on the subsequence of \( \{p_v\} \) at which \( J_\mu(p_v) \neq 0 \). The reason for this is that, if \( J_\mu(p_v) \neq 0 \), then \( J_\mu \) is (on the ray corresponding to) the Grassmann coordinates of \( T(X, p_v) \). Consequently \( c_v \in \{J_\mu(p_v), \cdots\} \) converges (on a subsequence of \( \{p_v\} \)) to a point \( b = (\cdots, b_2, \cdots) \) with the following properties (where \( \beta \) denotes the point of \( \mathbb{P}^{n-1} \) corresponding to \( b \)): \( (p, \beta) \in B_J(X); b_2 \neq 0 \) then \( b_2 \) is proportional to \( a \); \( b_2 \neq 0 \) for some \( \lambda \). Thus, given \( (p, \alpha) \) in \( \tau(X) \) we have constructed a point \( (p, \beta) \) in \( B_J(X) \). How many such points can exist? The construction depends not only on \( (p, \alpha) \) but also on \( a \). It is conceivable that for a different sequence \( \{q_v\} \), \( \{\lambda|b_2 \neq 0\} \) could change yielding another point of \( B_J(X, p) \) corresponding to \( (p, \alpha) \). Since the only possible variation is in \( \{\lambda|b_1 \neq 0\} \) and since there are only finitely many choices for the tuple \( \lambda \), we conclude there are at most finitely many points on \( B_J(X, p) \) for each point on \( \tau(X, p) \). Consequently (ii) and (iii) are equivalent and the first inequality is proved.

Suppose \( B_J(X, p) \) is finite. Since \( B_J(X) \rightarrow X \) is a proper modification, \( B_J(X, q) \) is finite for all \( q \) near \( p \) and there is a natural map \( \theta: (X', \mathcal{O}') \rightarrow (B_J(X), \mathcal{O}) \) commuting with the projections to \( X \) over a neighborhood of \( p \). \( \theta \) is proper and surjective. (This is the universal mapping property of
normalization [1, 46.20, p. 456]. \( \mathcal{J} \mathcal{O} \) is locally principal and \( \mathcal{J} \mathcal{O}' \) is the pull-back of \( \mathcal{J} \mathcal{O} \) induced by \( \theta \) so \( \mathcal{J} \mathcal{O}' \) is locally principal which proves (iii) implies (iv) and the second inequality.

We also use \( \theta \) to prove (iii) implies (v). Both \( C_4(X, p) \) and \( C(\cdot, \mathcal{O}) \) distribute across unions so we may assume \( X_p \) is irreducible and only one point \( p' \) of \( X' \) lies over \( p \). Hence, existence of \( \theta \) shows that only one point is in \( B_r(X, p) \). Consequently \( \tau(X, p) \) consists of one point. Hence, \( C_4(X, p) \) consists of a single plane of dimension \( r \). Since every component of \( C(X, p) \) has dimension \( r \) (proving (v) implies (iii)) and \( C_4(X, p) \subseteq C(X, p) \), it follows that they are equal. This proves that (iii) implies (v).

It remains to prove that (iv) implies (iii). If \( \mathcal{J} \mathcal{O}' \) is locally principal over \( p \) then there is a natural map \( \psi: (X', \mathcal{O}') \rightarrow (B_r(X, \mathcal{O})) \), which commutes with projections to \( X \) over a neighborhood of \( p \). \( \psi \) is proper and surjective. (This is the universal mapping property of blowing-up [3, p. 123].) Since \( X' \) has finite fiber over \( p \), \( B_r(X, p) \) is finite. Q.E.D.

**Corollary.** Suppose \( X_j \) denote the irreducible components of \( X \) at \( p \), \( I_p(X_j), I_p(X) \), is the ideal defining \( X \), resp. \( X_j, \) at \( p \) and \( J_j, J \), is the Jacobian ideal defined by \( I_p(X) \), resp. \( I_p(X_j) \). Then \( J \) becomes locally principal over \( p \) on the normalization of \( X \) if and only if \( J_j \) becomes principal over \( p \) on the normalization of \( X_j \) for all \( j \).

**Proof.** \( C_4(X, p) = \bigcup C_4(X_j, p) \). Q.E.D.

**Theorem 2.** Let \( R \) be the local ring of a point \( p \) on a reduced, pure \( r \)-dimensional analytic space \( X \). Suppose that the Jacobian ideal becomes locally principal over \( p \) on the normalization of \( X \). Then each minimal prime ideal \( P \) of \( G(R) \) determines at least one minimal prime ideal \( \mathfrak{p} \) of \( R \) such that \( P = \text{rad}(\text{Ker}(G(R) \rightarrow G(R/\mathfrak{p}))) \). Moreover, \( G(R)/\mathfrak{p} \) is a polynomial ring in \( r \) variables over \( \mathbb{C} \).

**Proof.** The hypothesis implies that \( C(X, p) = C_4(X, p) \) and is a finite union of \( r \)-dimensional subspaces of \( \mathbb{C}^n \). These planes are in one-to-one correspondence with the minimal primes of \( G(R) \). Let \( L \) be the plane corresponding to \( P \). Because \( C(\cdot, \mathcal{O}) \) distributes across unions, there is at least one irreducible component \( Z \) of \( X_p \) such that \( L \subseteq C(Z, p) \). Applying the inequality of Proposition 1 we conclude that \#\( \tau(Z, p) \) = 1 and \( L = C(Z, p) \). \( Z \) is determined by a minimal prime ideal of \( R \) and \( P \) is the radical of the kernel of \( G(R) \rightarrow G(R/\mathfrak{p}) \). \( G(R/\mathfrak{p}) \) has as reduction a ring of polynomials in \( r \) indeterminants because \( L \) is a plane of dimension \( r \). Since \( G(R) \rightarrow G(R/\mathfrak{p}) \) is surjective \( G(R)/\mathfrak{p} \) is isomorphic to the reduction of \( G(R/\mathfrak{p}) \). Q.E.D.
Corollary. If, in addition to the hypothesis of the theorem, \( R \) is a domain, then \( G(R) \) has as reduction an integral domain.

Proof. (0) is the only minimal prime of \( R \) so \( G(R) \) can have only one minimal prime. Q.E.D.

Examples. The locus of \( Y^2 - X^3 = 0 \) is an example showing that \( G(R) \) need not be a domain.

In general the correspondence between primes of \( G(R) \) and primes of \( R \) is one-to-many, e.g. the locus of \( Y(Y-X^2) = 0 \).

The locus of \( XY - Z^2 = 0 \) shows that assuming \( C(X, p) \) a union of planes and \( X_p \) irreducible does not insure that \( C(X, p) \) is irreducible.

Question. What condition together with irreducibility of \( X_p \) insures irreducibility of \( C(X, p) \)? The condition given here seems overly strong since it insures that \( C(X, p) \) is a plane, not just irreducible.

Recall the Whitney conditions [6, §8]. If \( X \) is an analytic space, \( Y \) is a manifold, and \( p \in X \cap Y \), \( X \) is said to be \( a \)-regular along \( Y \) at \( p \) if: whenever \( \{p_v\} \in X - Sg(X) \) with \( p_v \to p \) and \( T(X, p_v) \to T \) then \( T \supset T(Y, p) \). \( X \) is said to be \( b \)-regular along \( Y \) at \( p \) if: whenever \( \{p_v\} \in X - Sg(X) \), \( \{q_v\} \in Y \), \( \{c_v\} \in C \) with \( p_v \to p \), \( q_v \to q \), \( T(X, p_v) \to T \) and \( c_v(p_v - q_v) \to v \) then \( v \in T \). \( X \) is said to be \( a, b \)-regular along \( Y \) at \( p \) if \( X \) is both \( a \)-regular and \( b \)-regular along \( Y \) at \( p \).

We can see that \( J0' \) locally principal over \( p \) allows us to determine \( \lim T(X, p_v) = T \). For the limit to exist, infinitely many \( p_v \) must be on one component \( X_\gamma \) of \( X \) at \( p \) and \( T = C(X_\gamma, p) \). Thus \( a \)-regularity reduces to the simple property: every component of \( C(X, p) \) contains \( T(Y, p) \) at \( p \) if \( J0' \) is locally principal over \( p \). This suggests that we examine the question: Does \( J0' \) locally principal over \( p \) insure \( X \) is \( a, b \)-regular along \( Sg(X) \) at \( p \)?

First we observe that \( J0' \) locally principal over \( p \) implies that either \( p \notin Sg(X) \) or else \( p \in Sg(X) \) and \( \dim_p Sg(X) = r - 1 \). This is because the locus of \( J \) is \( \pi \) (locus of \( J0' \)). The latter either has dimension \( r - 1 \) at some point over \( p \) or is empty at every point over \( p \) and \( \pi \) preserves dimensions.

In what follows we sometimes require that \( Sg(X) \) be a manifold at \( p \). By this we mean that it is a manifold with the reduced structure, not that it is a manifold with the structure induced by \( J \).

A fundamental tool is the following proposition proved by John Stutz.

Proposition 3. Let \( X \) be a reduced analytic space of pure dimension \( r \) at \( p \) with \( p \in Sg(X) \) and \( Sg(X) \) a manifold at \( p \). Assume that \( \dim C(X, p) = r \).

Then \( X \) has a Puiseux series normalization at \( p \), i.e. there exists a ball \( D \subset C^r \) and holomorphic maps \( f_j : D \to X_j \) where \( X_j \) are the irreducible components of \( X \) at \( p \) such that

(a) \( f_j \) is a homeomorphism;

(b) there are coordinates \( (x), (y) \) in \( C^r \) and \( C^m \) (the ambient space for \( X \) at

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so that \( y(p) = 0 \) and \( f_j(x) = (x_1, \ldots, x_{r-1}, x_i^{d_j}, f_{r+1,j}(x), \ldots, f_{m,j}(x)) \), and the ball \( D \) and coordinates \( (x) \), \( (y) \) are independent of \( j \); and

(c) \( Y = C_{y_1, \ldots, y_{r-1}} \);

(d) if \( X_j \) contains \( Sg(X) \) at \( p \) then \( d_j \leq \text{order of } f_{k,j}(x) \) in \( x_r \), \( \forall k \).

**Proof.** [5, Propositions 4.2 and 4.6].

**Corollary.** With \( X \) as in the proposition the normalization of \( X \) is a manifold.

**Example.** The locus \( X \) of \( y^2 - x^2 (z - x) \) has singular set \( C_z \) and normalization a manifold but \( C_4(X, 0) \) has dimension three so the condition is not necessary.

**Theorem 4.** Let \( X \) be a reduced analytic space of pure dimension \( r \) at \( p \) with \( p \) in \( Sg(X) \) and \( Sg(X) \) a manifold at \( p \). Suppose that the Jacobian ideal of \( X \) becomes locally principal over \( p \) on the normalization of \( X \). Then \( X \) is \( a,b \)-regular along \( Sg(X) \) at \( p \) if and only if every irreducible component of \( X \) at \( p \) contains \( Sg(X) \) at \( p \). Any component of \( X \) which does not contain \( Sg(X) \) at \( p \) is a manifold at \( p \).

**Proof.** (We understand that Stutz has also proved this theorem in a paper of his to appear in the Amer. J. Math. so we give only an indication of the proof.) Using the Jacobian matrix of the mapping \( f_j \) of Proposition 3 it is easy to see that \( Sg(X_j) \subseteq Y \). If \( X_j \) does not contain \( Y \), \( \dim Sg(X_j) < r - 1 \). By an earlier remark \( Sg(X_j) = \emptyset \) which proves the last claim of the theorem. Now the only if part follows from Hironaka’s result [2, Corollary 6.2] that \( a,b \)-regularity implies equimultiplicity. The converse follows by careful analysis of the Jacobian matrix of the mapping \( f_j \) of Proposition 3. Q.E.D.

**Examples.** It is easy to use this theorem to construct an example in which \( J \) principalizes on the normalization but \( X \) is not \( a,b \)-regular along \( Sg(X) \) at \( p \). Zariski has given an example in which \( J \) becomes principal on the normalization but \( Sg(X) \) is not manifold at \( p \) [7, footnote 3, p. 987].

**Remarks.** In [5] Stutz proved, among other things, a number of the results of this paper under additional hypotheses. He proved the equivalence of (i) and (v) of Proposition 1 assuming that \( Sg(X) \) is a manifold of dimension \( r - 1 \) at \( p \) and \( \dim C_b(X, p) = r + 1 \). He proved Theorem 4 assuming \( p \) simple in \( Sg(X) \), \( Sg(X) \) a manifold of dimension \( r \) at \( p \), \( \dim C_b(X, p) = r \), and \( \dim C_b(X, p) = r + 1 \). We improve upon this result by applying Proposition 3 to the question of \( a,b \)-regularity directly rather than passing through the existence of wings as Stutz did. The assumptions used by Stutz insure that every component of \( X \) at \( p \) contains \( Sg(X) \) at \( p \) but he has told us the converse is not true. Where this paper extends part
of [5] most effectively is in the characterization of \( \dim C_4(X, p) = r \) by means of the Jacobian ideal (Zariski uses this technique to get a criterion of equisingularity in [7, Theorem 5.1, p. 987]). Not only does this technique yield better results, e.g. \( \dim C_4(X, p) = r \) always implies \( C_4(X, p) = C(X, p) \) and an avoidance in Theorem 4 of the cone \( C_5(X, p) \) which is difficult to compute, but it raises a number of interesting questions in the formal case.

**Question.** Suppose \( R = S/p \) where \( p \) is prime and \( S = k[[y_1, \ldots, y_n]] \) and \( R/J \) is regular and \( JR' \) is principal where \( R' \) is the integral closure of \( R \). Does it follow that \( R \) has a Puiseux series normalization? Is \( R' \) regular? Is the reduction of \( G(R) \) a domain? These results are true in the convergent case, yet the hypothesis and conclusion are punctual but the proofs are not. We hope to turn to these questions in a subsequent paper.

We are indebted to Abhyankar who proposed the question: What are the consequences of \( J \) locally principal over \( p \) on the normalization of \( X \)? and to the referee who suggested we examine the relation between some of these results and the paper of Stutz.

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