LOCAL BOUNDEDNESS AND CONTINUITY FOR A FUNCTIONAL EQUATION ON TOPOLOGICAL SPACES

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Abstract. It is known that the locally bounded solutions f of Cauchy's functional equation \( f(x) + f(y) = f(x+y) \) on the reals are necessarily continuous. We shall extend this result to the functional equation \( f(x) + g(y) = h(T(x, y)) \) on topological spaces.

1. Introduction. Let \( X, Y \) be topological spaces and let \( f: X \rightarrow \mathbb{R} \) (the reals), \( g: Y \rightarrow \mathbb{R} \), \( T: X \times Y \rightarrow \mathbb{R} \) and \( h: T(X \times Y) \rightarrow \mathbb{R} \) be functions satisfying the functional equation

\[
(1) \quad f(x) + g(y) = h(T(x, y))
\]

for all \( x \in X, y \in Y \). We shall give some sufficient topological assumptions on \( X \) and \( T \) so that the local boundedness and nonconstancy of \( f \) insure that \( g \) is continuous. The method was suggested by the work of J. Pfanzagl in his paper [6] generalizing a result of G. Darboux [2].

2. Main theorems.

Theorem 1. For equation (1), if each pair of points of \( X \) is contained in the continuous image of some connected and locally connected space (for instance, when \( X \) is connected and locally connected or when \( X \) is pathwise connected), \( T \) is continuous in each of its two variables and \( f \) is nonconstant and locally bounded from above (or from below) at each point of \( X \), then \( g \) is continuous on \( Y \).

Proof. Let \( a, b \in X \) be such that \( f(a) \neq f(b) \). There exist a connected and locally connected space \( \bar{X} \) and a continuous mapping \( \gamma: \bar{X} \rightarrow X \) such that \( a, b \in \gamma(\bar{X}) \). The functions \( \tilde{f} := f \circ \gamma \) and \( \tilde{T} \) with \( \tilde{T}(\tilde{x}, y) := T(\gamma(\tilde{x}), y) \) for \( \tilde{x} \in \bar{X}, y \in Y \), now satisfy the induced functional equation

\[
(\tilde{1}) \quad \tilde{f}(\tilde{x}) + g(\gamma(\tilde{x})) = h(\tilde{T}(\tilde{x}, y))
\]

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for all \( \bar{x} \in \mathcal{R} \), \( y \in Y \). The local boundedness of \( f \) passes to \( \bar{f} \) and the continuity of \( T \) in each variable passes to \( \bar{T} \). With this observation there is no loss of generality if we suppose from the very beginning that \( X \) is connected and locally connected.

Since \( X \) is connected and \( f \) is nonconstant on \( X \), \( f \) cannot be locally constant on \( X \) and there exists a point \( e \in X \) such that \( f \) is nonconstant on every neighbourhood of \( e \). As \( X \) is locally connected and \( f \) is locally bounded from above at \( e \) there exists an open connected neighbourhood \( U \) of \( e \) on which \( f \) is bounded from above. Thus \( f \) is nonconstant and bounded from above on the connected and locally connected set \( U \).

Let \( x_1, x_2 \in U \) be such that \( f(x_1) \neq f(x_2) \). It follows from equation (1) that \( T(x_1, y) \neq T(x_2, y) \) for all \( y \in Y \).

Let \( y_0 \in Y \) be arbitrarily given and we shall prove the continuity of \( g \) at \( y_0 \). We may suppose that \( t_1 := T(x_1, y_0) < T(x_2, y_0) =: t_2 \). By Lemma 1 in Pfanzagl [5] there exists a connected \( B \subseteq U \) such that \( T(B, y_0) = [t_1, t_2] \). Let \( \varepsilon > 0 \) be arbitrarily given. Since \( \sup f(B) < \infty \), there exists \( x_0 \in B \) such that \( f(x_0) \leq f(x) - \varepsilon \) for all \( x \in B \).

Let \( M := \{ y \in Y : T(x_0, y) < t_2 \} \). Then \( y_0 \in M \) and, as \( T(x_0, \cdot) \) is continuous on \( Y \), \( M \) is a neighbourhood of \( y_0 \).

For each \( y \in M \), \( T(x_0, y) \in [t_1, t_2] = T(B, y_0) \) and so there exists \( x \in B \) such that \( T(x_0, y) = T(x, y_0) \). Thus \( f(x_0) + g(y) = h(T(x_0, y)) = h(T(x, y_0)) = f(x) + g(y_0) \). As \( f(x_0) \geq f(x) - \varepsilon \) we have \( g(y_0) \leq g(y) + \varepsilon \).

Let \( x_3, x_4 \in B \) be arbitrarily chosen such that \( t_3 := T(x_3, y_0) < T(x_0, y_0) < T(x_4, y_0) =: t_4 \).

Let \( N := \{ y \in Y : T(x_3, y) < T(x_0, y_0) < T(x_4, y) \} \). Then \( y_0 \in N \) and, as \( T(x_3, \cdot) \) and \( T(x_4, \cdot) \) are continuous on \( Y \), \( N \) is a neighbourhood of \( y_0 \).

For each \( y \in N \), \( T(B, y) \) is an interval of \( R \) as \( B \) is connected and \( T(\cdot, y) \) is continuous. Furthermore, \( T(x_3, y) \) and \( T(x_4, y) \) are points of \( T(B, y) \) with \( T(x_3, y) < T(x_0, y_0) < T(x_4, y) \) and so \( T(x_0, y_0) \in T(B, y) \). Hence there exists \( x \in B \) such that \( T(x_0, y_0) = T(x, y) \). From this we have \( f(x_0) + g(y_0) = f(x) + g(y) \). As \( f(x_0) \geq f(x) - \varepsilon \) we have \( g(y_0) - \varepsilon \leq g(y) \).

\( M \cap N \) is then a neighbourhood of \( y_0 \) and \( g(y_0) - \varepsilon \leq g(y) \leq g(y_0) + \varepsilon \) for every \( y \in M \cap N \). This proves the continuity of \( g \) at \( y_0 \).

Remark 1. Lemma 1 in Pfanzagl [5] is given as: let \( X \) be a connected and locally connected Hausdorff space, \( \theta : X \rightarrow R \) a continuous map, then to any \( t_1, t_2 \in \theta(X) \) with \( t_1 < t_2 \), there exists a connected component \( B \) of \( \theta^{-1}([t_1, t_2]) \) such that \( \theta(B) = [t_1, t_2] \). The proof is based on a theorem of Wilder [7, p. 46, Theorem 3.8]. The assumption that \( X \) is Hausdorff is however not used and can be removed.

Corollary 1. If \( X \) is locally connected, \( T : X \times X \rightarrow R \) is continuous in each variable, \( f : X \rightarrow R \) is locally bounded from above (or from below) at
each point of $X$ and $h$ is any function on $T(X, X)$ satisfying the functional equation
\begin{equation}
 f(x) + f(y) = h(T(x,y))
\end{equation}
for all $x, y \in X$, then $f$ must be continuous on $X$.

**Proof.** For a point $a \in X$, if $f$ is locally constant at $a$ then $f$ is continuous at $a$. We may suppose now $f$ is not locally constant at $a$ and hence there exists an open connected neighbourhood $U$ of $a$ such that $f$ is bounded and nonconstant on $U$. We can apply Theorem 1 to the equation
\begin{equation}
 f(x) + f(y) = h(T(x,y))
\end{equation}
for all $x \in U$, $y \in X$ yielding the continuity of $f$ on $X$.

**Remark 2.** Corollary 1 is proved by Pfanzagl [6] under stronger assumptions on $X$—that $X$ is locally compact and locally connected Hausdorff.

**Theorem 2.** For equation (1), if each pair of points of $X$ is contained in some compact connected subset of $X$, $T$ is jointly continuous on the product space $X \times Y$ and $f$ is nonconstant and locally bounded from above on $X$ (or locally bounded from below on $X$), then $g$ is continuous on $Y$.

**Proof.** Similar to the argument given in the first paragraph in the proof of Theorem 1 we may suppose that $X$ is compact and connected. We note that $f$ is then bounded from above on every subset of $X$.

Let $y_0 \in Y$ and $\varepsilon > 0$ be arbitrarily given.

Since $f$ is nonconstant on $X$, for each $y \in Y$ the function $T(\cdot, y)$ is nonconstant on $X$ and $T(X, y)$ is a proper closed interval of $R$. Write $T(X, y_0) = [a, b]$ with $a < b$. Let $A = \{x \in X: T(x, y_0) = a\}$, $B = \{x \in X: T(x, y_0) = b\}$ and $C = \{x \in X: a < T(x, y_0) < b\}$. The sets $A$, $B$ and $C$ partitioned $X$ with $A$ and $B$ being closed in $X$ and therefore compact. Since $\sup f(C) < \infty$ there exists $x_0 \in C$ such that $f(x_0) \geq f(x) - \varepsilon$ for all $x \in C$.

We first let $M = \{y \in Y: a < T(x_0, y) < b\}$. Similar to the proof lines in Theorem 1, $M$ is seen to be a neighbourhood of $y_0$ and $g(y) \leq g(y_0) + \varepsilon$ for all $y \in M$.

Secondly, we let $N = \{y \in Y: T(x, y) < T(x_0, y_0) < T(x', y) \text{ for all } x \in A, x' \in B\}$. We proceed to show that $N$ is a neighbourhood of $y_0$.

For each $x \in A$ we have $T(x, y_0) = a \in [\infty, T(x_0, y_0)]$. $T$ is jointly continuous and so there exist neighbourhoods $U(x)$, $V_x(y_0)$ of $x$ and $y_0$ respectively such that $T(U(x), V_x(y_0)) \subseteq [\infty, T(x_0, y_0)]$. Now, because $A$ is compact, there exists a finite subset $A' \subseteq A$ such that $\bigcup \{U(x): x \in A'\} \supseteq A$. The finite intersection $V = \bigcap \{V_x(y_0): x \in A'\}$ is then a neighbourhood of $y_0$ and $T(A, V) \subseteq [\infty, T(x_0, y_0)]$. Similarly, there exists a
neighbourhood $W$ of $y_0$ such that $T(B, W) \subseteq ]T(x_0, y_0), \infty[$. Now $N \subseteq V \cap W$ and is a neighbourhood of $y_0$.

For each $y \in N$, $T(X, y)$ is an interval of $R$. The fact that $T(A, y) \subseteq ]-\infty, T(x_0, y_0)[$ and $T(B, y) \subseteq ]T(x_0, y_0), \infty[$ implies $T(x_0, y_0) \in T(C, y)$. Thus there exists $x \in C$ such that $T(x_0, y_0) = T(x, y)$. It follows that $f(x_0) + g(y_0) = f(x) + g(y)$. Since $f(x_0) \geq f(x) - \varepsilon$ we have $g(y_0) - \varepsilon \leq g(y)$.

$M \cap N$ is then a neighbourhood of $y_0$ and $g(y_0) - \varepsilon \leq g(y) \leq g(y_0) + \varepsilon$ for all $y \in M \cap N$. This proves the continuity of $g$ at $y_0$.

**Theorem 3.** For equation (1), if $X$ is connected, $T$ is continuous in each variable and $f$ is nonconstant and bounded on $X$ (from both sides), then $g$ is continuous on $Y$.

**Proof.** Let $y_0 \in Y$ and $\varepsilon > 0$ be arbitrarily given.

The nonconstancy of $f$ in equation (1) implies that $T(\cdot, y_0)$ is nonconstant. $T(X, y_0)$ is then a nondegenerated interval of $R$ and there exist $t_1, t_2 \in T(X, y_0)$ with $t_1 < t_2$. The set $B = \{ x \in X : T(x, y_0) \in ]t_1, t_2[ \}$ is mapped by $T(\cdot, y_0)$ onto $]t_1, t_2[$. Since $f$ is bounded from above on $B$ there exists $x_0 \in B$ such that $f(x_0) \geq f(x) - \varepsilon$ for all $x \in B$. If we set

$$M := \{ y \in Y : T(x_0, y) \in ]t_1, t_2[ \}$$

we see that $M$ is a neighbourhood of $y_0$. Furthermore for each $y \in M$, $T(x_0, y) \in ]t_1, t_2[ = T(B, y_0)$ and so there exists $x \in B$ such that $T(x_0, y) = T(x, y_0)$. It follows that $f(x_0) + g(y) = f(x) + g(y_0)$. As $f(x_0) \geq f(x) - \varepsilon$ we have $g(y) \leq g(y_0) + \varepsilon$.

The above argument applies to the functions $\tilde{f} = -f$, $\tilde{g} = -g$, and $\tilde{h} = -h$ satisfying again equation (1). Hence there exists a neighbourhood $\tilde{M}$ of $y_0$ such that $\tilde{g}(y) \leq \tilde{g}(y_0) + \varepsilon$ for all $y \in \tilde{M}$, i.e. $g(y_0) - \varepsilon \leq g(y)$.

On the neighbourhood $M \cap \tilde{M}$ we have $g(y_0) - \varepsilon \leq g(y) \leq g(y_0) + \varepsilon$ for every $y \in M \cap \tilde{M}$. This proves the continuity of $g$.

3. **Some examples.** The connectedness of $X$ is a common assumption in Theorems 1, 2 and 3. Its essentiality can be seen from the following example.

**Example 1.** We take $X = \{0, 1\}$ the discrete space $\subseteq R$, $Y = R$ the reals with the usual topology, $T(x+y) = x+y$, $f: X \to Y$ the natural inclusion map, $g = h: R \to R$ an additive function of the reals which is continuous at no place and leaving the rationals fixed. Obviously equation (1) is satisfied, $X$ is locally connected and compact, $T$ is jointly continuous and $f$ is bounded, nonconstant on $X$, while $g$ is continuous at no place.

However, connectedness of $X$ alone is not sufficient to give Theorems 1 and 2. This has been shown by C. Hipp who gave the following example.
Example 2 (by C. Hipp). Let $X = Y = R$, $Y$ endowed with the canonical topology $\tau$ on $R$ and $X$ endowed with the topology $\tau_1$ generated by $\tau$ and all subsets of $R$ containing the rational numbers $Q$. Then $(X, \tau_1)$ is connected. Let $\phi$ be a discontinuous (with respect to $\tau$) solution of the Cauchy equation

$$\phi(x) + \phi(y) = \phi(x + y) \quad \text{for all } x, y \in R.$$ 

As for each $x \in X$, $\phi$ is bounded on $\{(x) \cup Q\} \cap (x-1, x+1)$ which is a $\tau_1$ neighbourhood of $x$, we have the local boundedness of $\phi$ on $(X, \tau_1)$. The map $T$ with $T(x, y) = x + y$ is jointly continuous on $(X, \tau) \times (Y, \tau)$ and hence continuous on $(X, \tau_1) \times (Y, \tau)$. However $\phi$ is continuous on $(Y, \tau)$ at no place.

The local connectedness of $X$ for Corollary 1 is by no means redundant. We illustrate this by the following example.

Example 3. We take $X = \{n^{-1}: n = 1, 2, \cdots\} \cup \{0\}$ as a subspace of $R$, $T(x, y) = x + y \sqrt{2}$ on $X \times X$, $f(x) = 0$ if $x \neq 0$ and $f(0) = 1$, $h(n^{-1}) = h(n^{-1} \sqrt{2}) = 1$ for all $n = 1, 2, \cdots$ and $h(n^{-1} + m^{-1} \sqrt{2}) = 0$ for all $n, m = 1, 2, \cdots$ and $h(0) = 2$. Obviously, equation (2) is satisfied, $X$ fails to be locally connected at 0, $T$ is jointly continuous, $f$ is bounded on $X$ but fails to be continuous at 0.

Some uniqueness theorems concerning the continuous solutions of equations (1) and (2) are given in Ng [4] and Pfanzagl [5].

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