INTERSECTING UNIONS OF MAXIMAL CONVEX SETS

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Abstract. Hare and Kenelly have characterized the intersection of the maximal starshaped subsets of a set $S$, where $S$ is compact, simply connected and planar, and Sparks has solved the general problem for maximal $L_n$ sets. In this paper, a similar question is examined for unions of maximal convex sets: Let $S$ be a subset of $\mathbb{R}^2$, $\mathcal{C}$ the collection of all maximal convex subsets of $S$, and $\mathcal{N} = \{A \cup B : A, B \text{ distinct members of } \mathcal{C}\}$. Then $\bigcap \mathcal{N}$ is expressible as a union of three or fewer convex sets.

1. Intersecting unions of two maximal convex sets.

Lemma 1. Let $\mathcal{C}$ be any family of sets and $\mathcal{M} = \{A_1 \cup \cdots \cup A_k : A_1, \cdots, A_k \text{ distinct members of } \mathcal{C}\}$. Then $x \in \bigcap \mathcal{M}$ if and only if there are at most $k - 1$ members of $\mathcal{C}$ which fail to contain $x$.

Theorem 1. Let $\mathcal{C}$ be any collection of closed convex subsets of the plane and let $\mathcal{M} = \{A \cup B : A, B \text{ distinct members of } \mathcal{C}\}$. Then $\bigcap \mathcal{M}$ can be expressed as a union of three or fewer closed convex sets.

Proof. We assume that $\bigcap \mathcal{M}$ is not convex and consists of more than three points, and that $\mathcal{C}$ has at least three distinct members, for otherwise the result is trivial. We examine two cases.

Case 1. Assume that $\bigcap \mathcal{M}$ is three convex. That is, for $x, y, z$ in $\bigcap \mathcal{M}$, at least one of the corresponding segments lies in $\bigcap \mathcal{M}$. Since $\bigcap \mathcal{M}$ is closed, if it is connected, then by a theorem of Valentine [3], $\bigcap \mathcal{M}$ is expressible as a union of three or fewer closed convex sets having a nonempty intersection, completing the proof. If $\bigcap \mathcal{M}$ is not connected, then it has exactly two closed components, each of which is necessarily convex by the three convexity of $\bigcap \mathcal{M}$. This completes Case 1.

Case 2. If $\bigcap \mathcal{M}$ is not three convex, there are points $x, y, z$ in $\bigcap \mathcal{M}$ for which none of the corresponding segments lie in $\bigcap \mathcal{M}$. Thus there is some $A \cup B$ in $\mathcal{M}$ not containing all three segments. Assume $x, y \in A$, $z \in B \sim A$. Then $A$ is the only member of $\mathcal{C}$ not containing $z$ (by Lemma 1). Since $[x, y] \notin \bigcap \mathcal{M}$, there is some $C \cup D$ in $\mathcal{M}$ not containing $[x, y]$, and without loss of generality we may assume $x \in C \sim D$, $y \in D \sim C$, $z \in D$.

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Again by Lemma 1, $D$ is the only member of $\mathcal{G}$ not containing $x$, $C$ the only one not containing $y$. Also, since $y, z \in D$, $[y, z] \subseteq D$. Since $y \in A \sim C$, $A \neq C$, and $A \cup C$ belongs to $\mathcal{M}$. Thus $z \in C$ and $[x, z] \subseteq C$.

Moreover, for any member $E$ of $\mathcal{G}$ distinct from each of $A, C, D$, by Lemma 1, $E$ necessarily contains $x, y,$ and $z$.

We examine the following closed convex subsets of $\cap \mathcal{M}$. Define $A_0 = \bigcap \{E : E \in \mathcal{G}, E \neq A\}$, $C_0 = \bigcap \{E : E \in \mathcal{G}, E \neq C\}$, $D_0 = \bigcap \{E : E \in \mathcal{G}, E \neq D\}$. We will show that the only points of $\cap \mathcal{M}$ which fail to be in $A_0 \cup C_0 \cup D_0$ necessarily lie in $A \cap C \cap D \subseteq \ker(\cap \mathcal{M})$. It will then be easy to express $\cap \mathcal{M}$ as a union of three closed convex sets.

We begin by showing that $A \cap C \cap D$ is in $\conv(x, y, z)$. Let $p \in A \cap C \cap D$. If $[y, z]$ contains $p$, then $[y, p] \subseteq A$, $[p, z] \subseteq C$, and $[y, z] \subseteq A \cup C$. However $[y, z] \subseteq E$ for $E \neq A, C, D$. Also $[y, z] \subseteq D$, so $[y, z]$ would lie in $\cap \mathcal{M}$, a contradiction since none of the segments determined by $x, y, z$ lie in $\cap \mathcal{M}$.

A parallel argument shows that neither $[x, y]$ nor $[x, z]$ contains a point of $A_0 \cap C_0 \cap D_0$. Now for $p$ in $A \cap C \cap D$, if $[p, x]$ cut $[y, z]$ at $q$, then $[p, x] \subseteq A \cap C$, and $q \in A \cap C \cap D$ which is impossible by the preceding paragraph. Similarly $[p, y]$ cannot cut $[x, z]$, $[p, z]$ cannot cut $[x, y]$.

Also, $x \notin \conv(p, y, z)$ (for otherwise $x$ would lie in $D$), $y \notin \conv(p, x, z)$, and $z \notin \conv(p, x, y)$.

Hence $p$ must be interior to $\conv(x, y, z)$, and since $x, y, z \in E$ for every $E \neq A, C, D$, it follows that $p$ is in every member of $\mathcal{G}$ and in $\cap \mathcal{M}$.

Moreover, $p \in \ker(\cap \mathcal{M})$, for if $t \in \cap \mathcal{M}$, $t$ fails to belong to at most one $E$ in $\mathcal{G}$, so $[p, t]$ fails to lie in at most one member of $\mathcal{G}$, and $[p, t] \subseteq \cap \mathcal{M}$.

Now examine the sets $A_0, C_0, D_0$ defined previously. It is clear that each of these sets lies in $\cap \mathcal{M}$ by Lemma 1. For $u \in \cap \mathcal{M}$, either $u$ fails to lie in one of $A, C, D$ (and hence lies in one of $A_0, C_0, D_0$), or $u$ lies in $A \cap C \cap D$. Since $A \cap C \cap D \subseteq \ker(\cap \mathcal{M})$, the set $\conv((A \cap C \cap D) \cup A_0)$ is a subset of $\cap \mathcal{M}$. Thus each of the sets $A_1 = \overline{\conv((A \cap C \cap D) \cup A_0)}$, $C_1 = \overline{\conv((A \cap C \cap D) \cup C_0)}$, $D_1 = \overline{\conv((A \cap C \cap D) \cup D_0)}$ is a closed convex subset of the closed set $\cap \mathcal{M}$, and $\cap \mathcal{M} = A_1 \cup C_1 \cup D_1$, completing the proof.

REMARK. It is easy to find examples which show that the number three in Theorem 1 is best possible. (See Example 1 of this paper.)

Using Theorem 1, it is possible to prove the following.

**Theorem 2.** Let $S$ be planar, $\mathcal{G}$ the collection of all maximal convex subsets of $S$. Let $\mathcal{N} = \{A \cup B : A, B$ distinct members of $\mathcal{G}\}$. Then $\cap \mathcal{N}$ can be expressed as a union of three or fewer convex sets.
Proof. By an easy application of Theorem 1, for \( \mathcal{M} = \{ \text{cl } A \cup \text{cl } B: A, B \text{ distinct members of } \mathcal{C} \} \), \( \bigcap \mathcal{M} \) is a union of three or fewer closed convex sets \( S_i, i = 1, 2, 3 \).

Let \( M_i = S_i \cap (\bigcap \mathcal{N}) \), \( i = 1, 2, 3 \). If each \( M_i \) is convex, the proof is complete. Assume otherwise to reach a contradiction. Suppose for \( v, w \) in \( M_1 \), \( [v, w] \not\subseteq M_1 \). Then for some \( p \), \( v < p < w \), \( p \notin M_1 \). Therefore, there exist sets \( G, F \) in \( \mathcal{C} \) with \( p \notin G \cup F \). Without loss of generality, assume \( v \in G \sim F \), \( w \in F \sim G \).

If \( G, F \) are the only members of \( \mathcal{C} \), the proof is trivial. Otherwise, for every \( E \) in \( \mathcal{C} \sim \{ G, F \} \), \( [v, w] \subseteq E \) by Lemma 1. Thus \( p \in [v, w] \subseteq E \subseteq S \). Also, since \( p \in S_1 \), \( p \in \text{cl } G \cup \text{cl } F \), so assume \( p \in \text{cl } G \). For every \( x \) in \( G \), if \( [p, x] \subseteq S \), then the cone \( pG = \bigcup \{ [p, x]: x \text{ in } G \} \) would be a convex subset of \( S \) containing \( G \). But since \( G \) is maximal, this would imply that \( p \in G \), a contradiction. Hence for some \( x \) in \( G \), \( [x, p] \subseteq S \). Clearly such an \( x \) cannot lie on the line \( L(v, w) \) determined by \( v \) and \( w \), since \( [v, w] \subseteq S \) and \( [x, v] \subseteq S \).

For some \( y, x < y < p, y \notin S \). Since \( x, p \in \text{cl } G \), \( y \in \text{cl } G \), and since \( G \) is convex, \( y \) must lie on \( \text{bdry } G \). There is a supporting hyperplane \( H \) to \( \text{cl } G \) at \( y \), and \( H \) contains \( [x, p] \) since \( [y, p] \subseteq \text{cl } G \sim G \subseteq \text{bdry } G \). Note that this implies \( x \in \text{bdry } G \), and therefore \( p \) sees via \( S \) all points interior to \( G \). Clearly \( \text{int } G \not\subseteq S \) since \( x \notin L(v, w) \).

Consider the cone \( G_1 = p(\text{int } G) = \bigcup \{ [p, x]: x \in \text{int } G \} \). This is a convex subset of \( S \). If necessary, extend \( G_1 \) to a maximal convex subset \( G_2 \) of \( S \).

It is easy to see that \( w \notin G_2 \): Let \( U \) be any spherical neighborhood of \( x \) disjoint from the line \( L(w, y) \). Certainly \( U \) contains points of \( \text{int } G \), and for \( x_1 \) in \( U \cap \text{int } G \), \( y \in \text{conv} \{ x_1, p, w \} \). If \( w \) were in \( G_2 \), then \( y \in G_2 \subseteq S \), a contradiction since \( y \notin S \).

Now \( p \in G_2 \sim G \), so \( G \not\subseteq G_2 \) and \( G \cup G_2 \) is in \( \mathcal{N} \). Since \( w \in M_1 \subseteq \bigcap \mathcal{N} \), \( w \) must lie in \( G \cup G_2 \), but this is clearly impossible by the preceding paragraph. Hence our assumption is false, each \( M_i \) is convex, and \( \bigcap \mathcal{N} \) is a union of three or fewer convex sets.

2. The general case. It would be interesting to obtain analogues of Theorems 1 and 2 for unions of \( k \) convex sets. The following results, although for special cases, invite the conjecture that the appropriate bound is \( k(k+1)/2 \).

Theorem 3. Let \( \mathcal{C} \) be any collection of \( k+1 \) closed convex subsets of the plane and let \( \mathcal{M} = \{ A_1 \cup \cdots \cup A_k: A_1, \cdots, A_k \text{ distinct members of } \mathcal{C} \} \). Then \( \bigcap \mathcal{M} \) is expressible as a union of \( k(k+1)/2 \) or fewer closed convex sets. The result is best possible for all \( k \).

Proof. The proof is by induction. The result is trivial for \( k=1 \), and for \( k=2 \), the result is an immediate consequence of Theorem 1. Assume the theorem true for \( 2 < k-1 \) to prove for arbitrary \( k \).
Select any set $A$ in $\mathcal{C}$ and define subsets $P$, $Q$ of $\bigcap \mathcal{M}$ in the following manner. Let

$P = \{x: x \in A \text{ and } x \text{ fails to lie in exactly } k - 1 \text{ members of } \mathcal{C} - \{A\}\},$

$Q = \{x: x \text{ fails to lie in at most } k - 2 \text{ members of } \mathcal{C} - \{A\}\}.$

Note that $x \in Q$ if and only if either $x \in A$ and $x$ fails to lie in no more than $k-2$ members of $\mathcal{C}$ or $x \notin A$ and $x$ fails to lie in no more than $k-1$ members of $\mathcal{C}$. Using Lemma 1, it is clear that $P \cup Q = \bigcap \mathcal{M}$.

Examine the set $Q$. Now $\mathcal{C} - \{A\}$ is a collection of $k$ closed convex sets in the plane. Letting

$\mathcal{N} = \{B_1 \cup \cdots \cup B_{k-1}: B_1, \cdots, B_{k-1} \text{ distinct members of } \mathcal{C} - \{A\}\},$

by our induction hypothesis, $Q = \bigcap \mathcal{N}$ is expressible as a union of $(k-1)k/2$ or fewer closed convex sets.

Furthermore, any point of $P$ necessarily lies in exactly two members of $\mathcal{C}$, one of which is $A$. Letting $E_i = A \cap A_i$, $A_i$ in $\mathcal{C} - \{A\}$, $1 \leq i \leq k$, then

$P = \bigcup_{i=1}^{k} E_i.$

Hence $P \cup Q = \bigcap \mathcal{M}$ is a union of $(k-1)k/2 + k = k(k+1)/2$ or fewer closed convex sets, completing the proof.

**Example 1.** To see that the result in Theorem 3 is best possible, let $\mathcal{C}$ denote a collection of $k+1$ lines $L_i$, $1 \leq i \leq k+1$, every two intersecting and no three having a common point. Then the corresponding $\bigcap \mathcal{M}$ consists of exactly $k(k+1)/2$ isolated points.

In conclusion, we note that Example 1 reveals the "worst" case when $\mathcal{C}$ is any family of lines, for a proof paralleling that of Theorem 3 shows that the bound is again $k(k+1)/2$. The only additional step involves showing that for $A$ in $\mathcal{C}$, the corresponding $P$ may be represented as a union of $k$ or fewer convex sets: If more than $k$ convex sets were required, there would be at least $k+1$ distinct members of $\mathcal{C} - \{A\}$, each intersecting $A$ at a different point, and for $x$ in $A \cap (\bigcap \mathcal{M})$, $x$ would fail to lie in at least $k$ members of $\mathcal{C}$, contradicting Lemma 1. Thus $P$ has the desired representation and the result follows.

**References**


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