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AN ELEMENTARY PROOF OF THE MORSE-PALAIS LEMMA FOR BANACH SPACES

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Palais established the following generalization of the Morse lemma:

THEOREM (MORSE-PALAIS LEMMA [1]). *Let V be a real Banach space, $\mathcal{O}$ a convex neighborhood of the origin and let $f:\mathcal{O}\to\mathbf{R}$ be a C^{k+2} function ($k\geqq 1$) having the origin as a nondegenerate critical point with $f(0)=0$. Then there is a neighborhood U of the origin and a C^k diffeomorphism: $\Phi:U\to\mathcal{O}$ with $\Phi(0)=0$ and $(D\Phi)_0=\mathrm{id}_V$ (the identity map of V) such that for x in U, $f(\Phi(x))=\frac{1}{2}(D^2f)_0(x, x)$.*

Here and henceforth, the nonexplained notations are standard and are the same as in [1]. Palais' proof of the theorem draws on the theory of differential equations. We shall give of Palais' theorem a very elementary and short proof. Our proof, which only uses a direct application of the inverse mapping theorem, is even shorter and more elementary than Palais' proof of his theorem for the Hilbert space case [2]. The idea of our proof came from a close examination of Palais' proof for the Hilbert space case (loc. cit.)

We first observe that since 0 is a nondegenerate critical point of f, $(D^2f)_0$ is an isomorphism of V onto V^* and $(Df)_0=0$.

REMARK 1. The problem is obviously equivalent to that of establishing the existence of a C^k-isomorphism Ψ of an open neighborhood U_1 of 0 contained in $\mathcal{O}$ into $\mathcal{O}$, such that $\Psi(0)=0$, $(D\Psi)_0=\mathrm{id}_V$ and

$$f(y) = \tfrac{1}{2}(D^2f)_0(\Psi(y), \Psi(y)), \qquad y \text{ in } U_1.$$

For further use, we put $B=\frac{1}{2}(D^2f)_0$ and

$$E = \{h:h \in L(V, V) \text{ and } B(h(x))y = B(h(y))x,\ x \text{ and } y \text{ in } V\}.$$

Our proof of the Morse-Palais lemma depends on the following two simple lemmas.

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LEMMA 1. *E is a closed subspace of $L(V, V)$ and the map*

$$T: E \to L(V, V^*)$$

defined by $Th = B \circ h$ is an isomorphism of E onto $L_s(V, V^*)$.

PROOF. Let $Sh = B \circ h$, h in $L(V, V)$. Then S is an isomorphism of $L(V, V)$ onto $L(V, V^*)$ since B is an isomorphism. Clearly, T is the restriction of S to E. It remains to show that S maps E onto $L_s(V, V^*)$.

It is clear from the definition of E that S maps E into $L_s(V, V^*)$. Now let C be in $L_s(V, V^*)$ and let $Sh = C$, i.e., $B \circ h = C$. Then

$$B(h(x))y = C(x)y = C(y)x = B(h(y))x.$$

Hence h is in E. Q.E.D.

LEMMA 2. *Let $\theta: E \to L_s(V, V^*)$ be the map defined by*

$$(\theta h)(x)y = B(h(x))h(y).$$

Then θ is a C^∞-isomorphism of a neighborhood of id_V onto a neighborhood of B and $\theta(\mathrm{id}_V) = B$.

PROOF. By direct computation

$$[\theta(h + k) - \theta(h)](x)y = B(h(x))k(y) + B(k(x))h(y) + B(k(x))(k(y)).$$

Hence

$$(D\theta)_h(k)(x)y = B(h(x))k(y) + B(k(x))h(y).$$

Thus $D\theta$ is linear, and hence θ is C^∞. Furthermore

$$(D\theta)_{\mathrm{id}_V} k = 2B \circ k = 2Tk, \qquad k \text{ in } E.$$

By Lemma 1, $(D\theta)_{\mathrm{id}_V}$ is therefore an isomorphism, and hence, Lemma 2 follows from the inverse mapping theorem. Q.E.D.

PROOF OF THE MORSE-PALAIS LEMMA. By repeated applications of the fundamental theorem of calculus, and using the fact that $f(0) = 0$, $(Df)_0 = 0$, we have

$$f(y) = \int_0^1 \int_0^1 (D^2 f)_{tsy} t \, ds \, dt(y)y, \qquad y \text{ in } \mathcal{O}.$$

Put

$$G(y) = \int_0^1 \int_0^1 (D^2 f)_{tsy} t \, ds \, dt.$$

Then G is a C^k-map of $\mathcal{O}$ into $L_s(V, V^*)$ and $G(0) = \frac{1}{2}(D^2 f)_0 = B$.

By Lemma 2, there exists an open neighborhood W of B which is taken by the C^∞-map θ^{-1} onto an open neighborhood of id_V. Now G is in particular continuous, and hence maps an open neighborhood U' of 0

contained in $\mathcal{O}$ into W. Put

$$A(y) = \theta^{-1}(G(y)), \qquad \Psi(y) = A(y)y \quad \text{for } y \text{ in } U'.$$

Note that $A(y)$ is in $L(V, V)$. It is readily seen that A and Ψ are C^k on U', and $A(0)=\mathrm{id}_V$ and $\Psi(0)=0$.

Now

$$(D\Psi)_y = A(y) + (DA)_y y, \qquad (D\Psi)_0 = A(0) = \mathrm{id}_V.$$

Hence, by the inverse mapping theorem, Ψ maps an open neighborhood U_1 of 0 contained in U' (hence contained in $\mathcal{O}$) onto an open neighborhood U of 0 contained in $\mathcal{O}$. Furthermore

$$\begin{aligned} f(y) &= G(y)(y)y = \theta(A(y))(y)y \\ &= B(A(y))y(A(y))y = B(\Psi(y), \Psi(y)), \qquad y \text{ in } U_1. \end{aligned}$$

The theorem follows now from Remark 1 above.

References

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2. ———, *Morse theory on Hilbert manifolds*, Topology **2** (1963), 299–340. MR **28** #1633.

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