AN EXTREMAL PROPERTY OF SOME CAPACITARY MEASURES IN $E_n$

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Abstract. The capacitary measure on an arc of the circle is known (via conformal mapping) to be that measure of a class of measures which has the largest potential at certain points of the plane. Here it is shown that the analogous result is true in $E_n$.

1. Introduction. Let $E_n$ be Euclidean $n$ space, $n \geq 3$; denote vectors $(x_1, \ldots, x_n) \in E_n$ by $x$; and let $|x|$ be the norm of $x$.

Put $S = \{x \in E_n : |x| = 1\}$ and for a given positive $c$, $1 \leq c < \infty$, let $H(c)$ denote the class of those positive measures $\mu$ on $S$ of total mass 1 which satisfy

$$\phi(x) \leq c \quad \text{for each } x \in E_n.$$ 

Here $\phi(x) = \int_S |x-y|^{2-n} d\mu(y)$ is the potential of $\mu$.

Let $x \in E_n$, $|x| \neq 0$, $|x| \neq 1$. In this paper we characterize the measure in $H(c)$ which has the largest potential at $x$. The corresponding problem, where now $\phi$ is a logarithmic potential, occurs in $E_2$ in the study of the class $S^*$ of starlike univalent functions $f$ in $\{|x| < 1\}$, normalized by $f(0) = 0$, $f'(0) = 1$, and satisfying $|f| \leq e^{2\text{e}}$. Indeed, if $\phi$ is given, then the function $f$ defined by $\log(|f(z)|/|z|) = 2\phi(z)$, $|z| < 1$, $f'(0) = 1$, is in $S^*$. Conversely, given $f \in S^*$, $\frac{1}{2} \log(|f(z)|/|z|)$ is a potential whose associated measure is in $H(c)$. Hence the problem of characterizing the measure in $H(c)$ which has largest potential at $x$ is equivalent to finding the function $f$ in $S^*$ whose modulus is largest at $x$. Due to this equivalence, in $E_2$ the measure can be characterized by using conformal mapping. The measure has the same form in $E_n$. That is

**Theorem 1.** Let $x \in E_n$, $|x| \neq 0$, $|x| \neq 1$, $n \geq 3$. Let $\lambda$ be such that the cap $\{y : y \in S, |x-y| \leq \lambda\}$ has capacity $1/c$. Then if $e$ is the capacitary measure on this cap, the measure $ce$ has a greater potential at $x$ than any other measure in $H(c)$.

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With minor changes our method provides a new proof of the corresponding result in $E^2$.

We remark that T. J. Suffridge in $E^2$ [3] and J. L. Lewis in $E^3$ [2] have considered a related problem for potentials bounded below in $\{x:|x|\leq 1\}$ by a constant.

2. Several lemmas. Given $\alpha$, $0<\alpha\leq \pi$, let $C_\alpha=\{x \in S:x_1=\cos \alpha\}$. Let $\partial C_\alpha$ denote $\{x \in S:x_1=\cos \alpha\}$ and $\bar{C}_\alpha=\partial C_\alpha \cup C_\alpha$. Let $\mu_\alpha$ be the capacitary measure on $\bar{C}_\alpha$ and let $P_\alpha$ be its potential. Then

$$P_\alpha(x) = \int_{\partial C_\alpha} |x-y|^{2-n} d\mu_\alpha(y).$$

$P_\alpha$ is continuous on $E^2 \cup \{\infty\}$, harmonic in $E^2-C_\alpha$, 1 on $\bar{C}_\alpha$ and 0 at $\infty$, and is superharmonic in $E^2$. For fixed $c$, $1\leq c<\infty$, let $\alpha_0=\alpha_0(c)$ be such that $\Phi=cP_{\alpha_0}$ satisfies $\Phi(0)=1$, that is the potential of $c\mu_{\alpha_0}$ at 0 is 1. The existence of $\alpha_0$ is easily shown. We first prove

**Lemma 1.** If $\phi$ is the potential of a measure in $H(c)$ then for $0<\alpha\leq \pi$

$$\int_{\partial C_\alpha} \phi \ d\mu_\alpha \leq \int_{\bar{C}_\alpha} \Phi \ d\mu_\alpha.$$

**Proof.** The lemma is trivial if $\alpha=\alpha_0$ since $\Phi=c$ on $\bar{C}_{\alpha_0}$. If $\alpha>\alpha_0$ we interchange the order of integration in $\int_{\bar{C}_\alpha} \phi \ d\mu_\alpha$ and obtain for $\phi$ with $\mu$ in $H(c)$ that

$$\int_{\bar{C}_\alpha} \phi \ d\mu_\alpha = \int_S P_\alpha(y) \ d\mu(y) \leq \int_S \mu(y) = 1.$$

Similarly we have

$$\int_{\partial C_\alpha} \Phi \ d\mu_\alpha = \int_S cP_{\alpha_0}(y) \ d\mu_{\alpha_0} = \int_{\partial C_{\alpha_0}} c \ d\mu_{\alpha_0} = 1.$$

Next we note that

(i) $P_\alpha$ has continuous one sided normal derivatives on $C_\alpha$.

(ii) $\mu_\alpha(C_\theta)$ is a continuous function of $\theta$, $0<\theta\leq \pi$.

(i) is easily verified using the Kelvin transformation to map $\{x:|x|<1\}$ onto a halfspace (see Helms [1, pp. 36, 37]), and the reflection principle.

(ii) follows from the fact that $\mu_\alpha$ is capacitary measure on $\bar{C}_\alpha$. Let $\sigma$ denote Lebesgue measure on $S$ and $\sigma_n$ the Lebesgue measure of $S$. Differentiating (2.1) under the integral sign we find for $rx$, $0<r<1$, $|x|=1$, that

$$r^{2-n/2} \frac{\partial}{\partial r} [r^{n/2-1}P_\alpha(rx)]= \left(\frac{n}{2}-1\right) \int_{\partial C_\alpha} \frac{1-r^2}{|rx-y|^n} d\mu_\alpha(y).$$

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The last integral is a constant multiple, \((n/2 - 1)\sigma_n\), of the Poisson integral with boundary measure \(\mu_x\) evaluated at \(rx\). It follows from this fact that

\[
(n/2 - 1)^{-1}\sigma_n^{-1} \lim_{r \to 1} r^{2-n/2}(\partial/\partial r)[r^{n/2-1}P_a(rx)] = d\mu_x(x)/d\sigma
\]

almost everywhere with respect to \(\sigma\). Since (i) is true we deduce that \(\mu_x\) is absolutely continuous with respect to \(\sigma\) on \(C_x\) and thereupon that equality holds in (2.3) whenever \(x \in \partial C_x\). Since we also have (ii) it follows that \(\mu_x\) is absolutely continuous on \(C_x\).

We observe from the boundary values of \(P_a\) and (2.3) that the values of \(d\mu_x(x)/d\sigma\) depend only on the \(x_1\) coordinate of \(x\). Hence if we define \(\theta\) by \(x_1 = \cos \theta, 0 \leq \theta < \alpha\), and put \(h_x(\theta) = d\mu_x(x)/d\sigma, 0 \leq \theta < \alpha\), then \(h\) is well defined. With this notation we prove

**Lemma 2.** If \(0 < \alpha \leq \pi\), then \(h_x\) is a nonnegative nondecreasing function of \(\theta\) on \([0, \alpha]\) which is not everywhere 0.

**Proof.** Let \(x_0\) be that point on \(S\) with \(x_1 = 1, x_i = 0\) if \(i \neq 1\). We first show for \(0 < r < 1\) that

\[
P_a(ry) \leq P_a(rx)
\]

when \(x, y \in C_x\) and \(|x - x_0| < |y - x_0|\). To prove (2.4) let \(K\) be the hyperplane through the origin for which \(\bar{x} = y\), where \(\bar{x}\) denotes the reflection of \(x\) with respect to \(K\). Let \(G\) denote the halfspace bounded by \(K\) which contains \(x\). Define \(f\) in \(K \cup G\) by \(f(z) = P_a(z), z \in K \cup G\). Then \(f - P_a \leq 0\) on \(K \cup (C_x \cap G)\). Since \(f - P_a\) is bounded and harmonic in \(G - \bar{C}_a\) it follows from the maximum principle that \(f \leq P_a\) in \(G\) and consequently (2.4) is true.

Using (2.4) we get

\[
\int_0^s \frac{\partial}{\partial r} [r^{n/2-1}P_a(ry)] dr \leq \int_0^s \frac{\partial}{\partial r} [r^{n/2-1}P_a(rx)] dr
\]

for \(0 < s \leq 1\). Since equality holds for \(s = 1\) it follows that

\[
\lim_{r \to 1} (\partial/\partial r)[r^{n/2-1}P_a(rx)] \leq \lim_{r \to 1} (\partial/\partial r)[r^{n/2-1}P_a(ry)].
\]

Using (2.3) we conclude that \(h_x\) is nondecreasing on \([0, \alpha]\). Finally, \(h_x\) is somewhere positive since \(\mu_x(C_a) > 0\).

3. **Proof of Theorem 1.** It clearly suffices to prove Theorem 1 for points \(x\) of the form \(x = t x_0, 0 < t < \infty, t \neq 1\). Let \(Q\) be the Poisson kernel. Let \(\phi \in H(c), \phi \neq \Phi\), and suppose by way of contradiction that \(\Phi(t x_0) \leq \phi(t x_0)\). Put

\[
F(\theta) = \int_{C_\theta} Q(t x_0, y)[\Phi(y) - \phi(y)] d\sigma(y), \quad 0 < \theta \leq \pi.
\]
Then $\Phi(t x_0) - \phi(t x_0) = F(\pi)$, and so the following lemma immediately gives Theorem 1.

**Lemma 3.** If $\phi \neq \Phi$, then $F(\alpha) > 0$, $\alpha_0 \leq \alpha \leq \pi$.

**Proof.** We observe $F(\alpha) \geq 0$, $0 < \alpha < \alpha_0$, and $F(\alpha_0) > 0$, since $\Phi = c$ on $C_{\alpha_0}$. Hence if Lemma 3 is false there exists $\delta$, $\alpha_0 < \delta \leq \pi$, such that $F(\alpha) > 0$, $\alpha_0 \leq \alpha < \delta$, and $F(\delta) = 0$.

Define $Q$ on $[0, \pi]$ by $Q(\theta) = Q(t x_0, y)$ where $y$ satisfies $y_1 = \cos \theta$, and define $g_\delta(\theta) = h_\delta(\theta)/Q(\theta)$. Since $Q$ is strictly decreasing on $[0, \pi]$, Lemma 2 implies $g_\delta$ is nondecreasing and nonconstant on $[\alpha_0, \delta]$.

If $\alpha_0 < \beta < \delta$ then

\[
(3.2) \quad \int_{C_\beta} (\Phi - \phi) \, d\mu_\delta = \int_0^\beta g_\delta(\theta) \, dF(\theta) = F(\beta)g_\delta(\beta) - \int_0^\beta F(\theta) \, dg_\delta(\theta).
\]

The left-hand integral approaches

\[
\int_{C_\delta} (\Phi - \phi) \, d\mu_\delta \quad \text{as} \quad \beta \to \delta,
\]

since $\mu_\delta$ is absolutely continuous with respect to $\sigma$. Also

\[
\int_0^\delta g_\delta(\theta) \, |dF(\theta)| \geq F(\beta)g_\delta(\beta)
\]

and so $\lim_{\beta \to \delta} F(\beta)g_\delta(\beta) = 0$. Taking the limit as $\beta \to \delta$ in (3.2), we get

\[
\int_{C_\delta} (\Phi - \phi) \, d\mu_\delta = -\int_0^\delta F(\theta) \, dg_\delta(\theta).
\]

The right-hand side is negative since $g_\delta$ is nondecreasing and nonconstant on $[0, \delta]$ and $F$ is positive on $[\alpha_0, \delta]$. However by Lemma 1 the left-hand integral is nonnegative. We have reached a contradiction. Hence Lemma 3 is true.

4. **Remark.** We note that $\Phi$ also solves the corresponding minimum problem. That is

\[
(4.1) \quad \Phi(-t x_0) < \min_{|x|=t} \phi(x),
\]

when $\phi$ is a potential of a measure on $H(c)$, $\phi$ is not a rotation of $\Phi$, $0 < t < \infty$, and $t \neq 1$. The proof is exactly the same except we substitute $-Q(-t x_0, y)$ for $Q(t x_0, y)$ in (3.1). It appears likely that (4.1) is also true for $t = 1$, although we have not been able to prove this.
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