A THEOREM ON THE RESTRICTION OF TYPE I REPRESENTATIONS OF A GROUP TO CERTAIN OF ITS SUBGROUPS

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Abstract. Theorem. Let $G$ be a separable locally compact group and $H$ a closed subgroup such that $G/H$ is finite. Let $\pi$ be a Type I representation of $G$. Then $\pi|H$ is Type I.

The purpose of this note is to prove the following theorem.

Theorem 1. Let $G$ be a separable locally compact group and $H$ a closed subgroup such that $G/H$ is finite. Let $\pi$ be a Type I representation of $G$. Then $\pi|H$ is Type I.

The proof of this theorem is surprisingly intricate. Note that since $G$ is separable, we may assume that $\pi$ acts on a separable Hilbert space.

If $R$ is a von Neumann algebra, let $P(R)$ be the lattice of all projections in $R$.

Lemma 2. Let $R$ be an Abelian von Neumann algebra, $F$ a finite group and $\varphi(a)$ a homomorphism of $F$ into the group of $*$-automorphisms of $R$. Then there exists a sequence of nonzero, mutually orthogonal projections $\{P_n\}$ in $P(R)$ such that: (1) $\sum_{n=1}^{\infty} P_n = I$; (2) if $a \in F$, $Q \in P(R)$, $Q \leq P_n$ (for some $n$), then either $\varphi(a)(Q) = Q$ or $\varphi(a)(Q)$ is orthogonal to $P_n$.

Proof. Let $a \in F$. Recall (Kallman [2]) that there exists a projection $P(a, 1)$ in $P(R)$ such that if $Q \in P(R)$, $Q \leq P(a, 1)$, then $\varphi(a)(Q) = Q$, and if $Q \geq I - P(a, 1)$, then there exists $Q' \in P(R)$, $0 \neq Q' \leq Q$, such that $\varphi(a)(Q') \perp Q'$. Hence, an easy application of Zorn's lemma shows that there exists a sequence of nonzero projections $\{P(a, n)\}_{n \geq 2}$ such that $\varphi(a)(P(a, n)) \perp P(a, n)$ and $\sum_{n \geq 2} P(a, n) = I - P(a, 1)$. Now consider the nonzero elements of $[\inf_{a \in F} P(a, i_a) | i_a \geq 1]$. These are a countable set of projections in $P(R)$, whose sum is $I$, and a little thought shows that they satisfy (2). Q.E.D.

Received by the editors October 23, 1972.


Key words and phrases. Operator theory, von Neumann algebras, group representations.

1 Work performed under the auspices of the U.S. Atomic Energy Commission.
Lemma 3. Let $R$ be a von Neumann algebra with no finite Type I summands, $F$ a finite group, $a \rightarrow \varphi(a)$ a representation of $F$ into the group of $*$-automorphisms of $R$ which leave $\text{Cent } R$ pointwise fixed. Then there exists a noncentral operator $T$ in $R$ which is fixed by every $\varphi(a)$ $(a \in F)$.

Proof. Since $R$ has no finite Type I summands, there exists a sequence \[ P_n \mid n \geq 1 \subset P(R) \] such that $P_m \perp P_n$ $(m \neq n)$ and $C(P_n) = C(P_m) = C(m, n \geq 1)$, where $C(P_m)$ is the central support of $P_m$. Consider $Q_m = \sum_{a \in F} \varphi(a)(P_m)$. If some $Q_m$ is noncentral, then we are done, for each $Q_m$ is clearly invariant under $F$. Hence, suppose $Q_m$ central for all $m \geq 1$. Notice that if $C \in P(\text{Cent } R)$, $C' \leq C$, then the support of $Q_m \cdot C'$ is $C'$. In particular, the support of $Q_m = C$. Also, $\|Q_m \cdot C'\| \geq \|P_m \cdot C'\| = 1$ since $P_m \cdot C'$ is a nonzero projection in $R$. Hence, $Q_m \geq C$. But $Q_m \rightarrow 0$ in the strong operator topology. This is nonsense since $C$ is fixed and nonzero. Hence, some $Q_m$ is noncentral. Q.E.D.

Lemma 4. For every integer $n \geq 2$, there exists an $n \times n$ selfadjoint matrix with real entries which has zero diagonal, which has 1 as an eigenvalue with multiplicity 1, and has $-1/(n-1)$ as an eigenvalue with multiplicity $n-1$.

Proof. We merely write down such a matrix for every $n \geq 2$. Let $T_n = (t_{ij})$, where $t_{ii} = 0$ $(1 \leq i \leq n)$ and $t_{ij} = 1/(n-1)$ $(i \neq j)$. Compute that $(1, 1, \cdots, 1)$ is an eigenvector with eigenvalue 1, and that $(-1, 1, 0, \cdots, 0), (0, -1, 1, 0, \cdots, 0), \cdots, (0, \cdots, -1, 1)$ are eigenvectors with eigenvalue $-1/(n-1)$.

Since these $n$ vectors are linearly independent, the lemma is proved. Q.E.D.

Lemma 5. Let $R$ be a factor and $A \subset R$ an Abelian von Neumann subalgebra which has no minimal projections. Then for every integer $m \geq 1$, there exists a Type $I_m$ subfactor $S \subset R$ such that $A \cap S$ is maximal Abelian in $S$.

Proof. First of all, note that for every $m \geq 1$, there exist $m$ nonzero mutually orthogonal equivalent projections in $A$ with sum $I$. This is clear if $R$ is a Type $\text{III}_\infty$ factor, for choose any $m$ nonzero mutually orthogonal projections with sum $I$. Any two such projections are equivalent (remember our Hilbert space is separable) in a Type $\text{III}_\infty$ factor.

Next, suppose $R$ is a $\text{I}_\infty$ or a $\text{II}_\infty$ factor. First, suppose that there exist $Q_1, Q'_1 \in P(A), Q_2 \perp Q'_1, Q_1$ and $Q'_1$ infinite. If $Q_2 = I - Q_1$, then $Q_2 \in P(A)$, $Q_2 \perp Q_1$, and $Q_2$ is infinite. Hence, $Q_2$ is equivalent to $Q_1$. If no such pair $Q_1, Q'_1$ exists, then $I = \sum_{m \geq 1} P_m$, where the $P_m$ are finite, mutually orthogonal projections in $P(A)$. Then an elementary exercise using the semifinite trace on $R$ shows that there exist subsets $Z_1, Z_2 \subset \mathbb{Z}^+$ such that $\mathbb{Z}^+ = \mathbb{Z}_1 \cup \mathbb{Z}_2$, $Z_1 \cap Z_2 = \emptyset$, $Q_1 = \sum_{m \in Z_1} P_m$ is infinite, and $Q_2 = \sum_{m \in Z_2} P_m$ is infinite.
Hence, in either case, there exist \( Q_1, Q_2 \in P(A) \), \( Q_1 \) and \( Q_2 \) infinite, \( Q_1 \) is equivalent to \( Q_2 \), and \( Q_1 + Q_2 = I \). Consider \( Q_2 R Q_2 \) and \( Q_2 A Q_2 \). \( Q_2 R Q_2 \) is a \( I_\infty \) or \( II_\infty \) factor and \( Q_2 A Q_2 \) is an Abelian von Neumann subalgebra with no minimal projections. Hence, as above, there exist two infinite orthogonal equivalent projections \( Q'_2 \) and \( Q'_3 \) in \( Q_2 A Q_2 \) such that \( Q_2 = Q'_2 + Q'_3 \). But then \( Q_1, Q_2, Q_3 \) are three infinite orthogonal equivalent projections in \( R \) with sum \( I \). Continuing this argument, one sees that for every \( m \geq 1 \), there exist \( m \) nonzero mutually orthogonal equivalent projections in \( A \) with sum \( I \).

If \( R \) is a \( II_1 \) factor, it suffices to show that there exists a projection \( Q \in P(A) \) such that \( \text{trace}(Q) = 1/m \). There exists a nonzero projection \( P \in P(A) \) such that \( \text{trace}(P) < 1/m \) since \( A \) has no minimal projections. By Zorn's lemma choose a maximal chain \([Q_a]\) of elements of \( P(A) \), ordered by inclusion, such that \( \text{trace}(Q_a) \leq 1/m \). Then \( Q = \sup_{a} Q_a \in P(A) \). We claim that \( \text{trace}(Q) = 1/m \). If not, then \( \text{trace}(Q) < 1/m \). Then there exists a nonzero \( Q' \in P(A) \) such that \( \text{trace}(Q') < 1/m - \text{trace}(Q) \) and \( Q' \leq I - Q \). \( Q' \) exists since \( A \) has no minimal projections. But then \([Q_a] \cup [Q + Q']\) is a properly larger chain than \([Q_a]\), and \( \text{trace}(Q + Q') \leq 1/m \). This is a contradiction. Hence, \( \text{trace}(Q) = 1/m \).

\( R \) cannot be a finite Type I factor, for any Abelian von Neumann subalgebra of \( R \) is generated by its minimal projections.

Therefore, let \( Q_1, \cdots, Q_m \) be \( m \) nonzero orthogonal equivalent projections in \( A \) with sum \( I \). Let \( W_{11} = Q_1 \) and let \( W_{ij} \) \( (2 \leq j \leq m) \) be a partial isometry whose initial projection is \( Q_j \) and whose final projection is \( Q_j \). Set \( W_{ji} = W_{ji}^* \) and \( W_{ij} = W_{ji} \cdot W_{ij} \). Then \( W_{ij} \) is a partial isometry with initial projection \( Q_j \) and final projection \( Q_j \). \( W_{ij} \cdot W_{kl} = \delta_{jk} \cdot W_{il} \), and \( W_{ji} = Q_j \). One checks easily that the linear span of the \( W_{ij} \) \((1 \leq i, j \leq m)\) is a selfadjoint algebra \( S \) which is a Type \( I_m \) factor.

Finally, \( S \cap A \rightarrow \) the linear span of \( Q_1, \cdots, Q_m \). But this linear span is a maximal Abelian selfadjoint subalgebra of \( S \). Hence, \( S \cap A \) actually equals this linear span. \( \text{QED} \).

**Lemma 6.** Let \( R \) be a factor, \( F \) a finite group, and \( a \rightarrow \varphi(a) \) a homomorphism of \( F \) into the group of *-automorphisms of \( R \). Let \( A \) be the von Neumann subalgebra of \( R \) consisting of those elements of \( R \) which are left fixed by every \( \varphi(a) \) \((a \in F) \). Suppose \( A \) is Abelian. Then \( A \) is generated by its minimal projections.

**Proof.** It suffices to show that \( A \) has at least one nonzero minimal projection. Suppose not. Let \( f \) be the number of elements in \( F \). Choose a large integer \( m \) such that \( m - 1 > f \). By Lemma 5, there exists an \( m \times m \) matrix subalgebra \( S \) of \( R \) such that we may identify \( A \cap S \) with the algebra of all diagonal matrices in \( S \). Choose \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m \) and let
$A \in S \cap A$ be the matrix with $\lambda_1, \cdots, \lambda_n$ as diagonal elements. Let $T$ be the $m \times m$ matrix described in Lemma 4. In a natural manner, $T \in S$. Define a third skew-adjoint matrix $M = (s_{ij})$ in $S$ by $s_{ij} = (\lambda_i - \lambda_j) (i \neq j)$. One computes that $A \cdot M - M \cdot A = T$. For any $a \in F$, note that $A \cdot \varphi(a)(M) - \varphi(a)(M) \cdot A = \varphi(a)(T)$.

Since $A$ is the fixed point set of the $\varphi(a)$ ($a \in F$), we have that

$$A \cdot \sum_{a \in F} \varphi(a)(M) = \sum_{a \in F} \varphi(a)(M) \cdot A,$$

or

$$\sum_{a \in F} \varphi(a)T = 0.$$  

But this is not true. Let $T = T^+ - T^-$ be the decomposition of $T$ into its positive and negative parts. $\|T^+\| = 1$ and $\|T^-\| = 1/(m-1)$ by Lemma 4. Hence, $\|\sum_{a \in F} \varphi(a)(T^+)\| \geq \|T^+\| \geq 1$. But $\|\sum_{a \in F} \varphi(a)(T^-)\| \leq 1$. Hence, $\sum_{a \in F} \varphi(a)(T)$ is nonzero. Hence, $A$ must have at least one nonzero minimal projection. Q.E.D.

Let $Z$ be an Abelian von Neumann algebra on the Hilbert space $K$. Recall that there exists a standard Borel space $\Xi$, a Borel measure $\mu$ on $\Xi$, and a measurable field of separable Hilbert spaces $\xi \rightarrow K_\xi$ on $\Xi$ such that $K = \int_\Xi K_\xi d\mu(\xi)$ and $Z = L_\infty(\Xi, d\mu)$. Let $A$ be a separable $C^*$-algebra and $a \rightarrow \pi(a)$ a $*$-representation of $A$ on $K$. We say that $\pi$ is nondegenerate if the identity operator $I$ is in the strong closure of $\pi(A)$.

**Lemma 7.** Let $\pi(A)$ commute with $Z$. Then for every $\xi \in \Xi$, there exists a $*$-representation $\pi_\xi(\cdot)$ of $A$ on $K_\xi$ such that $\xi \rightarrow \pi_\xi(a)$ is measurable for every $a \in A$ and $\pi(a) = \int_\Xi \pi_\xi(a) d\mu(\xi)$. There exists a $\mu$-null Borel set $N \subset \Xi$ such that for $\xi \in \Xi - N$, $\pi_\xi(\cdot)$ is nondegenerate.

**Proof.** The existence of the $\pi_\xi(\cdot)$ follows easily from Dixmier [1, Théorème 1, p. 167], since $\pi(A)$ has a countable dense subalgebra over the complex rationals. That $\pi_\xi(\cdot)$ is nondegenerate for $\mu$-almost all $\xi$ follows easily from Dixmier [1, Proposition 4, p. 162], since $\pi(A)$ is nondegenerate. Q.E.D.

Henceforth, we can (and do) assume that $N = \emptyset$.

**Proposition 8.** Suppose that $\pi(\cdot)$ is Type I. Then there exists a $\mu$-null Borel set $N \subset \Xi$ such that for $\xi \in \Xi - N$, $\pi_\xi(\cdot)$ is Type I.

**Proof.** Let $R$ be the von Neumann algebra generated by $Z$ and $\pi(A)$. $Z$ and $R(\pi)$ are both Type I and commute. Hence, $R$ is Type I since any finite collection of commuting Type I von Neumann algebras generate a Type I von Neumann algebra. There exists a measurable field $\xi \rightarrow R(\xi)$ of von Neumann algebras on $K_\xi$ such that $R = \int_\Xi R(\xi) d\mu(\xi)$. Let

$$Z_1 = \int_\Xi Z_1(\xi) d\mu(\xi), \quad Z_2 = \int_\Xi Z_2(\xi) d\mu(\xi), \cdots$$

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be a countable subset of $\mathbb{Z}$ which is strongly dense in $\mathbb{Z}$. We may assume that $Z_n(\xi)$ is a scalar for all $n \geq 1$ and $\xi \in \Xi$. Since the $Z_n (n \geq 1)$ and $\pi(A)$ generate $R$, there exists a $\mu$-null Borel set $N_1 \subseteq \Xi$ such that for $\xi \in \Xi - N_1$, $R(\xi)$ is generated by $Z_n(\xi)$ $(n \geq 1)$ and $\pi_\xi(A)$ (see Dixmier [1, Théorème 1, p. 178]). Since $R$ is Type I, there exists a $\mu$-null Borel set $N_2$ such that for $\xi \in \Xi - N_2$, $R(\xi)$ is Type I (Dixmier [1, Proposition 7, p. 190]). Let $N = N_1 \cup N_2$. Then for $\xi \in \Xi - N$, $R(\xi)$ is Type I. Also $\pi_\xi(A)$ generates $R(\xi)$ since each $Z_n(\xi)$ is a scalar and $\pi_\xi(A)$ is nondegenerate. Hence, for $\xi \in \Xi - N$, $\pi_\xi(\cdot)$ is Type I. Q.E.D.

**Proof of Theorem 1.** Let $e = g_1, \ldots, g_n$ be coset representatives for $G/H$. $H' = \bigcap_{1 \leq i < n} g_i^{-1} H g_i$ is normal in $G$. $G/H'$ is finite by Poincaré’s lemma. If we can show that $\pi|H'$ is Type I, then $\pi|H$ is Type I by the main theorem of Kallman [3]. Hence, we may assume that $H$ is normal in $G$.

Consider $R(\pi)$. Since $\pi$ is Type I, we may assume that $R(\pi)'$ is Abelian. Let us apply Lemma 2 to $\text{Cent } R(\pi|H)$ and the finite group of $*$-automorphisms induced by $G/H$. Take some $P_n \cdot P_n \cdot R(\pi) \cdot P_n$ is Type I and is generated by $P_n \cdot \pi(g) \cdot P_n (g \in G)$. Now either $P_n \cdot \pi(g) \cdot P_n = 0$ or $\pi(g)$ commutes with every projection $Q \in P_n \cdot \text{Cent } R(\pi|H)$. Let $G' = \{g \in G : P_n \cdot \pi(g) \cdot P_n \neq 0\}$. $H \subset G'$, $g \rightarrow P_n \cdot \pi(g)$ is a unitary representation of $G'$, and the $P_n \cdot \pi(g)$ ($g \in G'$) generate $P_n \cdot R(\pi) \cdot P_n$. Now $R(\pi|H)$ is Type I if and only if $P_n \cdot R(\pi|H)$ is Type I for every $n \geq 1$.

Hence, we may suppose that $H$ is normal in $G$, that $R(\pi)'$ is Abelian, and that $\text{Cent } R(\pi|H) = \text{Cent } R(\pi)$.

There exists a standard Borel space $\Xi$, a positive Borel measure $\mu$ on $\Xi$, a measurable field of separable Hilbert spaces $\xi \rightarrow K_\xi$, and a measurable field of strongly continuous unitary representations $\xi \rightarrow \pi_\xi$ of $G$ on $K_\xi$ such that $K = \int_\Xi K_\xi d\mu(\xi)$, $\pi(g) = \int_\Xi \pi_\xi(g) d\mu(\xi)$, and $\text{Cent } R(\pi|H) = L^\infty(\Xi, d\mu)$. Since $\text{Cent } R(\pi|H) \subseteq \text{Cent } R(\pi)$, Proposition 8 implies that we may assume each $\pi_\xi$ is Type I. $\pi|H$ will be Type I if we can show that each $\pi_\xi|H$ is Type I since $\text{Cent } R(\pi|H) = L^\infty(\Xi, d\mu)$. Hence, we may suppose $\pi|H$ is a factor.

$G/H$ has a natural homomorphism into the group of $*$-automorphisms of $R(\pi|H)'$ whose fixed point set is $R(\pi)' = \text{Cent } R(\pi)$. By Lemma 6, $\text{Cent } R(\pi)$ is generated by its minimal projections.

Choose minimal orthogonal projections $Q_1, Q_2, \ldots \in P(\text{Cent } R(\pi))$ such that $\sum_{m \geq 1} Q_m = I$, $R(\pi|H)$ is Type I if and only if $Q_m \cdot R(\pi|H)$ is Type I for $m \geq 1$. $Q_m \cdot R(\pi)$ is a Type I factor since $Q_m$ is a minimal projection in $\text{Cent } R(\pi)$. $Q_m \cdot R(\pi|H)$ is also a factor. Hence, suppose that both $R(\pi)$ and $R(\pi|H)$ are factors and $R(\pi)'$ is the scalars.

Since $H$ is normal in $G$, $G/H$ has a natural homomorphism into the group of $*$-automorphisms of $R(\pi|H)'$. Since $R(\pi)'$ is the scalars, the fixed point set of $G/H$ acting on $R(\pi|H)'$ is the scalars. Hence, Lemma 3 implies
that $R(\pi|H)'$ is a finite Type I factor. Hence, $R(\pi|H)$ is a Type I factor. Q.E.D.

BIBLIOGRAPHY


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