PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 1, September 1973

A THEOREM ON THE RESTRICTION OF TYPE I REPRESENTATIONS OF A GROUP TO CERTAIN OF ITS SUBGROUPS[1]

ROBERT R. KALLMAN

ABSTRACT. THEOREM. *Let G be a separable locally compact group and H a closed subgroup such that G/H is finite. Let π be a Type* I *representation of G. Then $\pi|H$ is Type* I.

The purpose of this note is to prove the following theorem.

THEOREM 1. *Let G be a separable locally compact group and H a closed subgroup such that G/H is finite. Let π be a Type* I *representation of G. Then π/H is Type* I.

The proof of this theorem is surprisingly intricate. Note that since G is separable, we may assume that π acts on a separable Hilbert space.

If $\boldsymbol{R}$ is a von Neumann algebra, let $P(\boldsymbol{R})$ be the lattice of all projections in $\boldsymbol{R}$.

LEMMA 2. *Let $\boldsymbol{R}$ be an Abelian von Neumann algebra, F a finite group and $a\to\varphi(a)$ a homomorphism of F into the group of *-automorphisms of $\boldsymbol{R}$. Then there exists a sequence of nonzero, mutually orthogonal projections $[P_n]$ in $P(\boldsymbol{R})$ such that*: (1) $\sum_{n=1} P_n=I$; (2) *if $a\in F$, $Q\in P(\boldsymbol{R})$, $Q\leqq P_n$ (for some n), then either $\varphi(a)(Q)=Q$ or $\varphi(a)(Q)$ is orthogonal to P_n.*

PROOF. Let $a\in F$. Recall (Kallman [2]) that there exists a projection $P(a, 1)$ in $P(\boldsymbol{R})$ such that if $Q\in P(\boldsymbol{R})$, $Q\leqq P(a, 1)$, then $\varphi(a)(Q)=Q$, and if $Q\leqq I-P(a, 1)$, then there exists $Q'\in P(\boldsymbol{R})$, $0\neq Q'\leqq Q$, such that $\varphi(a)(Q')\perp Q'$. Hence, an easy application of Zorn's lemma shows that there exists a sequence of nonzero projections $[P(a, n)|n\geqq 2]$ such that $\varphi(a)(P(a, n))\perp P(a, n)$ and $\sum_{n\geqq 2} P(a, n)=I-P(a, 1)$. Now consider the nonzero elements of $[\inf_{a\in F} P(a, i_a)|i_a\geqq 1]$. These are a countable set of projections in $P(\boldsymbol{R})$, whose sum is I, and a little thought shows that they satisfy (2). Q.E.D.

Received by the editors October 23, 1972.
AMS (MOS) subject classifications (1970). Primary 22D10, 46L10.
Key words and phrases. Operator theory, von Neumann algebras, group representations.
[1] Work performed under the auspices of the U.S. Atomic Energy Commission.

LEMMA 3. *Let $\boldsymbol{R}$ be a von Neumann algebra with no finite Type* I *summands, F a finite group, $a \to \varphi(a)$ a representation of F into the group of *-automorphisms of $\boldsymbol{R}$ which leave* Cent $\boldsymbol{R}$ *pointwise fixed. Then there exists a noncentral operator T in $\boldsymbol{R}$ which is fixed by every $\varphi(a)$ $(a \in F)$.*

PROOF. Since $\boldsymbol{R}$ has no finite Type I summands, there exists a sequence $[P_n | n \geqq 1] \subset P(\boldsymbol{R})$ such that $P_m \perp P_n$ $(m \neq n)$ and $C(P_n) = C(P_n) = C(m, n \geqq 1)$, where $C(P_m)$ is the central support of P_m. Consider $Q_m = \sum_{a \in F} \varphi(a)(P_m)$. If some Q_m is noncentral, then we are done, for each Q_m is clearly invariant under F. Hence, suppose Q_m central for all $m \geqq 1$. Notice that if $C' \in P(\text{Cent } \boldsymbol{R})$, $C' \leqq C$, then the support of $Q_m \cdot C'$ is C'. In particular, the support of $Q_m = C$. Also, $\|Q_m \cdot C'\| \geqq \|P_m \cdot C'\| = 1$ since $P_m \cdot C'$ is a nonzero projection in $\boldsymbol{R}$. Hence, $Q_m \geqq C$. But $Q_m \to 0$ in the strong operator topology. This is nonsense since C is fixed and nonzero. Hence, some Q_m is noncentral. Q.E.D.

LEMMA 4. *For every integer $n \geqq 2$, there exists an $n \times n$ selfadjoint matrix with real entries which has zero diagonal, which has* 1 *as an eigenvalue with multiplicity* 1, *and has $-1/(n-1)$ as an eigenvalue with multiplicity $n-1$.*

PROOF. We merely write down such a matrix for every $n \geqq 2$. Let $T_n = (t_{ij})$, where $t_{ii} = 0$ $(1 \leqq i \leqq n)$ and $t_{ij} = 1/(n-1)$ $(i \neq j)$. Compute that $(1, 1, \cdots, 1)$ is an eigenvector with eigenvalue 1, and that $(-1, 1, 0, \cdots, 0)$, $(0, -1, 1, 0, \cdots, 0)$, $\cdots$, $(0, \cdots, -1, 1)$ are eigenvectors with eigenvalue $-1/(n-1)$.

Since these n vectors are linearly independent, the lemma is proved. Q.E.D.

LEMMA 5. *Let $\boldsymbol{R}$ be a factor and $A \subset \boldsymbol{R}$ an Abelian von Neumann subalgebra which has no minimal projections. Then for every integer $m \geqq 1$, there exists a Type I_m subfactor $\boldsymbol{S} \subset \boldsymbol{R}$ such that $A \cap \boldsymbol{S}$ is maximal Abelian in $\boldsymbol{S}$.*

PROOF. First of all, note that for every $m \geqq 1$, there exist m nonzero mutually orthogonal equivalent projections in A with sum I. This is clear if $\boldsymbol{R}$ is a Type III_∞ factor, for choose any m nonzero mutually orthogonal projections with sum I. Any two such projections are equivalent (remember our Hilbert space is separable) in a Type III_∞ factor.

Next, suppose $\boldsymbol{R}$ is a I_∞ or a II_∞ factor. First, suppose that there exist $Q_1, Q_1' \in P(A)$, $Q_1 \perp Q_1'$, Q_1 and Q_1' infinite. If $Q_2 = I - Q_1$, then $Q_2 \in P(A)$, $Q_2 \perp Q_1$, and Q_2 is infinite. Hence, Q_2 is equivalent to Q_1. If no such pair Q_1, Q_1' exists, then $I = \sum_{m \geqq 1} P_m$, where the P_m are finite, mutually orthogonal projections in $P(A)$. Then an elementary exercise using the semifinite trace on $\boldsymbol{R}$ shows that there exist subsets $Z_1, Z_2 \subset Z^+$ such that $Z^+ = Z_1 \cup Z_2$, $Z_1 \cap Z_2 = \varnothing$, $Q_1 = \sum_{m \in Z_1} P_m$ is infinite, and $Q_2 = \sum_{m \in Z_2} P_m$ is infinite.

Hence, in either case, there exist $Q_1, Q_2 \in P(A)$, Q_1 and Q_2 infinite, Q_1 is equivalent to Q_2, and $Q_1+Q_2=I$. Consider $Q_2\boldsymbol{R}Q_2$ and Q_2AQ_2. $Q_2\boldsymbol{R}Q_2$ is a I_∞ or II_∞ factor and Q_2AQ_2 is an Abelian von Neumann subalgebra with no minimal projections. Hence, as above, there exist two infinite orthogonal equivalent projections Q_2' and Q_3' in Q_2AQ_2 such that $Q_2=Q_2'+Q_3'$. But then Q_1, Q_2', Q_3' are three infinite orthogonal equivalent projections in $\boldsymbol{R}$ with sum I. Continuing this argument, one sees that for every $m \geqq 1$, there exist m nonzero mutually orthogonal equivalent projections in A with sum I.

If $\boldsymbol{R}$ is a II_1 factor, it suffices to show that there exists a projection $Q \in P(A)$ such that $\operatorname{trace}(Q)=1/m$. There exists a nonzero projection $P \in P(A)$ such that $\operatorname{trace}(P)<1/m$ since A has no minimal projections. By Zorn's lemma choose a maximal chain $[Q_\alpha]$ of elements of $P(A)$, ordered by inclusion, such that $\operatorname{trace}(Q_\alpha) \leqq 1/m$. Then $Q=\sup_\alpha Q_\alpha \in P(A)$. We claim that $\operatorname{trace}(Q)=1/m$. If not, then $\operatorname{trace}(Q)<1/m$. Then there exists a nonzero $Q' \in P(A)$ such that $\operatorname{trace}(Q')<1/m-\operatorname{trace}(Q)$ and $Q' \leqq I-Q$. Q' exists since A has no minimal projections. But then $[Q_\alpha] \cup [Q+Q']$ is a properly larger chain than $[Q_\alpha]$, and $\operatorname{trace}(Q+Q') \leqq 1/m$. This is a contradiction. Hence, $\operatorname{trace}(Q)=1/m$.

$\boldsymbol{R}$ cannot be a finite Type I factor, for any Abelian von Neumann subalgebra of $\boldsymbol{R}$ is generated by its minimal projections.

Therefore, let $Q_1, \cdots, Q_m$ be m nonzero orthogonal equivalent projections in A with sum I. Let $W_{11}=Q_1$ and let W_{1j} $(2 \leqq j \leqq m)$ be a partial isometry whose initial projection is Q_j and whose final projection is Q_1. Set $W_{j1}=W_{1j}^*$ and $W_{ij}=W_{i1} \cdot W_{1j}$. Then W_{ij} is a partial isometry with initial projection Q_j and final projection Q_i, $W_{ij} \cdot W_{kl}=\delta_{jk} \cdot W_{il}$, and $W_{jj}=Q_j$. One checks easily that the linear span of the W_{ij} $(1 \leqq i, j \leqq m)$ is a selfadjoint algebra $\boldsymbol{S}$ which is a Type I_m factor.

Finally, $\boldsymbol{S} \cap A \supset$ the linear span of $Q_1, \cdots, Q_m$. But this linear span is a maximal Abelian selfadjoint subalgebra of $\boldsymbol{S}$. Hence, $\boldsymbol{S} \cap A$ actually equals this linear span. Q.E.D.

LEMMA 6. *Let $\boldsymbol{R}$ be a factor, F a finite group, and $a \to \varphi(a)$ a homomorphism of F into the group of *-automorphisms of $\boldsymbol{R}$. Let A be the von Neumann subalgebra of $\boldsymbol{R}$ consisting of those elements of $\boldsymbol{R}$ which are left fixed by every $\varphi(a)$ $(a \in F)$. Suppose A is Abelian. Then A is generated by its minimal projections.*

PROOF. It suffices to show that A has at least one nonzero minimal projection. Suppose not. Let f be the number of elements in F. Choose a large integer m such that $m-1>f$. By Lemma 5, there exists an $m \times m$ matrix subalgebra $\boldsymbol{S}$ of $\boldsymbol{R}$ such that we may identify $A \cap \boldsymbol{S}$ with the algebra of all diagonal matrices in $\boldsymbol{S}$. Choose $0<\lambda_1<\lambda_2<\cdots<\lambda_m$ and let

$A \in S \cap A$ be the matrix with $\lambda_1, \cdots, \lambda_n$ as diagonal elements. Let T be the $m \times m$ matrix described in Lemma 4. In a natural manner, $T \in S$. Define a third skew-adjoint matrix $M=(m_{ij})$ in S by $m_{ii}=0$ $(1 \leqq i \leqq m)$ and $m_{ij}=t_{ij}/(\lambda_i-\lambda_j)$ $(i \neq j)$. One computes that $A \cdot M - M \cdot A = T$. For any $a \in F$, note that $A \cdot \varphi(a)(M) - \varphi(a)(M) \cdot A = \varphi(a)(T)$.

Since A is the fixed point set of the $\varphi(a)$ $(a \in F)$, we have that

$$A \cdot \sum_{a \in F} \varphi(a)(M) = \sum_{a \in F} \varphi(a)(M) \cdot A, \quad \text{or} \quad \sum_{a \in F} \varphi(a)T = 0.$$

But this is not true. Let $T=T^+-T^-$ be the decomposition of T into its positive and negative parts. $\|T^+\|=1$ and $\|T^-\|=1/(m-1)$ by Lemma 4. Hence, $\|\sum_{a \in F} \varphi(a)(T^+)\| \geqq \|T^+\| \geqq 1$. But $\|\sum_{a \in F} \varphi(a)(T^-)\| \leqq f \cdot \|T^-\| < 1$. Hence, $\sum_{a \in F} \varphi(a)(T)$ is nonzero. Hence, A must have at least one nonzero minimal projection. Q.E.D.

Let Z be an Abelian von Neumann algebra on the Hilbert space K. Recall that there exists a standard Borel space Ξ, a Borel measure μ on Ξ, and a measurable field of separable Hilbert spaces $\xi \to K_\xi$ on Ξ such that $K = \int_{\Xi}^{\oplus} K_\xi \, d\mu(\xi)$ and $Z = L^\infty(\Xi, d\mu)$. Let A be a separable C^*-algebra and $a \to \pi(a)$ a $*$-representation of A on K. We say that π is nondegenerate if the identity operator I is in the strong closure of $\pi(A)$.

Lemma 7. *Let $\pi(A)$ commute with Z. Then for every $\xi \in \Xi$, there exists a $*$-representation $\pi_\xi(\cdot)$ of A on K_ξ such that $\xi \to \pi_\xi(a)$ is measurable for every $a \in A$ and $\pi(a) = \int_{\Xi}^{\oplus} \pi_\xi(a) \, d\mu(\xi)$. There exists a μ-null Borel set $N \subset \Xi$ such that for $\xi \in \Xi - N$, $\pi_\xi(\cdot)$ is nondegenerate.*

Proof. The existence of the $\pi_\xi(\cdot)$ follows easily from Dixmier [**1**, Théorème 1, p. 167], since $\pi(A)$ has a countable dense subalgebra over the complex rationals. That $\pi_\xi(\cdot)$ is nondegenerate for μ-almost all ξ follows easily from Dixmier [**1**, Proposition 4, p .162], since $\pi(A)$ is nondegenerate. Q.E.D.

Henceforth, we can (and do) assume that $N = \varnothing$.

Proposition 8. *Suppose that $\pi(\cdot)$ is Type* I. *Then there exists a μ-null Borel set $N \subset \Xi$ such that for $\xi \in \Xi - N$, $\pi_\xi(\cdot)$ is Type* I.

Proof. Let R be the von Neumann algebra generated by Z and $\pi(A)$. Z and $R(\pi)$ are both Type I and commute. Hence, R is Type I since any finite collection of commuting Type I von Neumann algebras generate a Type I von Neumann algebra. There exists a measurable field $\xi \to R(\xi)$ of von Neumann algebras on K_ξ such that $R = \int_{\Xi}^{\oplus} R(\xi) \, d\mu(\xi)$. Let

$$Z_1 = \int_{\Xi}^{\oplus} Z_1(\xi) \, d\mu(\xi), \qquad Z_2 = \int_{\Xi}^{\oplus} Z_2(\xi) \, d\mu(\xi), \cdots$$

be a countable subset of $\boldsymbol{Z}$ which is strongly dense in $\boldsymbol{Z}$. We may assume that $Z_n(\xi)$ is a scalar for all $n \geqq 1$ and $\xi \in \Xi$. Since the Z_n $(n \geqq 1)$ and $\pi(A)$ generate $\boldsymbol{R}$, there exists a μ-null Borel set $N_1 \subseteq \Xi$ such that for $\xi \in \Xi - N_1$, $\boldsymbol{R}(\xi)$ is generated by $Z_n(\xi)$ $(n \geqq 1)$ and $\pi_\xi(A)$ (see Dixmier [**1**, Théorème 1, p. 178]). Since $\boldsymbol{R}$ is Type I, there exists a μ-null Borel set N_2 such that for $\xi \in \Xi - N_2$, $\boldsymbol{R}(\xi)$ is Type I (Dixmier [**1**, Proposition 7, p. 190]). Let $N = N_1 \cup N_2$. Then for $\xi \in \Xi - N$, $\boldsymbol{R}(\xi)$ is Type I. Also $\pi_\xi(A)$ generates $\boldsymbol{R}(\xi)$ since each $Z_n(\xi)$ is a scalar and $\pi_\xi(A)$ is nondegenerate. Hence, for $\xi \in \Xi - N$, $\pi_\xi(\cdot)$ is Type I. Q.E.D.

PROOF OF THEOREM 1. Let $e = g_1, \cdots, g_n$ be coset representatives for G/H. $H' = \bigcap_{1 \leqq i \leqq n} g_i \cdot H \cdot g_i^{-1}$ is normal in G. G/H' is finite by Poincaré's lemma. If we can show that $\pi|H'$ is Type I, then $\pi|H$ is Type I by the main theorem of Kallman [**3**]. Hence, we may assume that H is normal in G.

Consider $\boldsymbol{R}(\pi)$. Since π is Type I, we may assume that $\boldsymbol{R}(\pi)'$ is Abelian. Let us apply Lemma 2 to Cent $\boldsymbol{R}(\pi|H)$ and the finite group of *-automorphisms induced by G/H. Take some P_n. $P_n \cdot \boldsymbol{R}(\pi) \cdot P_n$ is Type I and is generated by $P_n \cdot \pi(g) \cdot P_n (g \in G)$. Now either $P_n \cdot \pi(g) \cdot P_n = 0$ or $\pi(g)$ commutes with every projection $Q \in P_n \cdot$ Cent $\boldsymbol{R}(\pi|H)$. Let $G' = [g \in G | P_n \cdot \pi(g) \cdot P_n \neq 0]$. $H \subset G'$, $g \to P_n \cdot \pi(g)$ is a unitary representation of G', and the $P_n \cdot \pi(g)$ $(g \in G')$ generate $P_n \cdot \boldsymbol{R}(\pi) \cdot P_n$. Now $\boldsymbol{R}(\pi|H)$ is Type I if and only if $P_n \cdot \boldsymbol{R}(\pi|H)$ is Type I for every $n \geqq 1$.

Hence, we may suppose that H is normal in G, that $\boldsymbol{R}(\pi)'$ is Abelian, and that Cent $\boldsymbol{R}(\pi|H) \subset$ Cent $\boldsymbol{R}(\pi)$.

There exists a standard Borel space Ξ, a positive Borel measure μ on Ξ, a measurable field of separable Hilbert spaces $\xi \to \boldsymbol{K}_\xi$, and a measurable field of strongly continuous unitary representations $\xi \to \pi_\xi$ of G on $\boldsymbol{K}_\xi$ such that $\boldsymbol{K} = \int_\Xi^\oplus \boldsymbol{K}_\xi \, d\mu(\xi)$, $\pi(g) = \int_\Xi^\oplus \pi_\xi(g) \, d\mu(\xi)$, and Cent $\boldsymbol{R}(\pi|H) = L^\infty(\Xi, d\mu)$. Since Cent $\boldsymbol{R}(\pi|H) \subset$ Cent $\boldsymbol{R}(\pi)$, Proposition 8 implies that we may assume each π_ξ is Type I. $\pi|H$ will be Type I if we can show that each $\pi_\xi|H$ is Type I since Cent $\boldsymbol{R}(\pi|H) = L^\infty(\Xi, d\mu)$. Hence, we may suppose $\pi|H$ is a factor.

G/H has a natural homomorphism into the group of *-automorphisms of $\boldsymbol{R}(\pi|H)'$ whose fixed point set is $\boldsymbol{R}(\pi)' =$ Cent $\boldsymbol{R}(\pi)$. By Lemma 6, Cent $\boldsymbol{R}(\pi)$ is generated by its minimal projections.

Choose minimal orthogonal projections $Q_1, Q_2, \cdots \in P($Cent $\boldsymbol{R}(\pi))$ such that $\sum_{m \geqq 1} Q_m = I$. $\boldsymbol{R}(\pi|H)$ is Type I if and only if $Q_m \cdot \boldsymbol{R}(\pi|H)$ is Type I for $m \geqq 1$. $Q_m \cdot \boldsymbol{R}(\pi)$ is a Type I factor since Q_m is a minimal projection in Cent $\boldsymbol{R}(\pi)$. $Q_m \cdot \boldsymbol{R}(\pi \mid H)$ is also a factor. Hence, suppose that both $\boldsymbol{R}(\pi)$ and $\boldsymbol{R}(\pi|H)$ are factors and $\boldsymbol{R}(\pi)'$ is the scalars.

Since H is normal in G, G/H has a natural homomorphism into the group of *-automorphisms of $\boldsymbol{R}(\pi|H)'$. Since $\boldsymbol{R}(\pi)'$ is the scalars, the fixed point set of G/H acting on $\boldsymbol{R}(\pi|H)'$ is the scalars. Hence, Lemma 3 implies that $\boldsymbol{R}(\pi|H)'$ is a finite Type I factor. Hence, $\boldsymbol{R}(\pi|H)$ is a Type I factor. Q.E.D.

Bibliography

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957. MR **20** #1234.

2. Robert R. Kallman, *A generalization of free action*, Duke Math. J. **36** (1969), 781–789. MR **41** #838.

3. ———, *Certain topological groups are type* I, Bull. Amer. Math. Soc. **76** (1970), 404–406. MR **41** #385.

Department of Mathematics, University of California, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87544

Current address: Department of Mathematics, University of Florida, Gainesville, Florida 32601