ON $L^p$ NORMS AND THE EQUIMEASURABILITY OF FUNCTIONS
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Abstract. For measurable functions $f$ and $g$, necessary and sufficient conditions are given for the equality of certain $L^p$ norms of $f$ and $g$ to imply that $f$ and $g$ are equimeasurable.

Two Lebesgue measurable functions $f$ and $g$ defined on $I = [0, 1]$ are said to be equimeasurable (or are rearrangements of each other) if they have the same distribution function, that is, if $m$ denotes Lebesgue measure on $I$ and

$$D_f(y) = m\{t: |f(t)| > y\} \quad (y \geq 0)$$

then $D_f = D_g$. It is very well known that the equimeasurability of $f$ and $g$ ensures that the $L^p$ norms of $f$ and $g$ are equal for every $p$, $1 \leq p \leq \infty$. Here, as usual, the $L^p$ norm of $f$ is defined by

$$||f||_p = \begin{cases} \left( \int_0^1 |f(t)|^p \, dt \right)^{1/p} & 1 \leq p < \infty, \\ \text{ess sup} |f(t)|, & p = \infty. \end{cases}$$

In this paper we determine necessary and sufficient conditions in order that the equality of certain $L^p$ norms of $f$ and $g$ will ensure that $f$ and $g$ are equimeasurable. More precisely we have the following results:

Theorem 1. Suppose $f$ and $g$ are essentially bounded functions on $I$ and let $P = P(f, g) = \{p : p \geq 1 : ||f||_p = ||g||_p\}$. If $P$ contains a sequence of distinct points $\{p_n\}$ with the property that $\sum_1^\infty (1/p_n) = \infty$ then $f$ and $g$ are equimeasurable, and in particular, $P = \{p : p \geq 1\}$. Conversely, given a sequence $\{p_n\}$ with $p_n \geq 1$ and $\sum_1^\infty (1/p_n) < \infty$, there exist bounded measurable functions $f$ and $g$ on $I$ which are not equimeasurable, but $||f||_{p_n} = ||g||_{p_n}$, $n = 1, 2, \ldots$.

Corollary. If $P$ has a finite limit point, in particular if $P$ is uncountable, then $f$ and $g$ are equimeasurable.

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Theorem 2. Suppose $f$ and $g$ are measurable functions on $I$ for which $\|f\|_p$ and $\|g\|_p$ are finite if and only if $1 \leq p \leq p_\infty < \infty$. If $P = P(f, g) = \{p: 1 \leq p \leq p_\infty, \|f\|_p = \|g\|_p\}$ contains a sequence of distinct points $\{p_n\}$ such that $\sum_{n=1}^{\infty} (p_\infty - p_n) = \infty$, then $f$ and $g$ are equimeasurable. Conversely, given $p_\infty, 1 < p_\infty < \infty$, and a sequence $\{p_n\}, 1 \leq p_n \leq p_\infty$, such that $\sum_{n=1}^{\infty} (p_\infty - p_n) < \infty$, there exist measurable functions $f$ and $g$ defined on $I$ with finite $L^p$ norms if and only if $1 \leq p \leq p_\infty$ which are not equimeasurable but $\|f\|_p = \|g\|_p$ if and only if $p \in \{p_n\}$.

The proof we give of Theorem 1 is an elementary application of the Hahn-Banach Theorem and the Theorem of Müntz. A second proof can be given which follows the line of our proof of Theorem 2, and which yields a slightly stronger conclusion in the “converse” part of Theorem 1, namely, the equality of the $L^p$ norms of $f$ and $g$ if and only if $p \in \{p_n\}$. However, since Theorem 1 appears to be the most useful for applications, it seems desirable to sacrifice the additional strength in favour of a simple proof.

For the proof we require the following well-known result (see, for example, [1, Lemma 3.3.2, p. 182]):

$$
\|f\|_p^p = p \int_0^\infty y^{p-1} D_f(y) \, dy \quad (1 \leq p < \infty).
$$

Proof of Theorem 1. (Sufficiency) By considering, if necessary, $cf$ and $cg$ where $c$ is a constant, we may assume that both $f$ and $g$ are essentially bounded in absolute value by 1. Now if $C(I)$ denotes, as usual, the space of continuous functions on $I$ with the $L^\infty$ norm, then

$$
L(h) = \int_I h(t) (D_f(t) - D_g(t)) \, dt \quad (h \in C(I))
$$

defines a bounded linear functional on $C(I)$ which by (1) vanishes on the functions $h_p(t) = t^p, p \in \{p_n-1\}$. Now by Müntz’ theorem (see, for example, [2, p. 305]) the functions $h_p, p > 0$, belong to the closure of the set $\{h_{p_n}^{-1}: n = 1, 2, \cdots\}$ and hence $L$ vanishes on $h_p, p > 0$, by continuity of $L$. But then by the theorem of dominated convergence it follows that

$$
L(h_0) = \int_0^1 \lim_{p \to 0+} t^p (D_f - D_g)(t) \, dt = \lim_{p \to 0+} L(h_p) = 0
$$

and hence we must have $L = 0$, that is $D_f = D_g$.

(Necessity) Suppose that $\{p_n\}$ is given with $\sum_{n=1}^{\infty} (1/p_n) < \infty$. Then, again by Müntz’ theorem, the set of functions $S = \{h_p: p = 0 \text{ or } p = p_n, n = 1, 2, \cdots\}$ is not dense in $C(I)$, and hence there is a bounded linear functional $L, \|L\| \leq 1$, on $C(I)$ which vanishes on $S$ but $L(h_{p_0}) \neq 0$ for some $p_0 > 1$. 

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Now, according to the representation theorem for bounded linear functionals on $C(I)$, there exist nonnegative, nonincreasing functions $\alpha, \beta$, defined on $I$ such that

$$L(h) = \int_0^1 h(t) \, d(\alpha - \beta)(t) \quad (h \in C(I))$$

and $\alpha(1) = \beta(1) = 0$. Moreover, we have $\alpha \leq 1$ and $\beta \leq 1$ since $\alpha(0) + \beta(0) = \text{Total Variation of } (\alpha - \beta) = \|L\| \leq 1$. Hence, if we define $f$ and $g$ to be, respectively, the right continuous inverse of $\alpha$ and $\beta$, then $D_f = \alpha$, $D_g = \beta$ and if $p \geq 1$ integration by parts yields

$$L(h_p) = \int_0^1 t^p \, d(D_f - D_g) = -p \int_0^1 t^{p-1}(D_f - D_g) \, dt = \|g\|^p - \|f\|^p$$

so that $\|f\|^p = \|g\|^p$ for $p \in \{p_n : n = 1, 2, \ldots\}$ but $\|f\|^p \neq \|g\|^p$ and in particular, $D_f \neq D_g$.

The proof of Theorem 2 requires the following lemma:

**Lemma.** Let $F(s)$ be an analytic function in the strip $S = \{ s : -\infty < a < \Re s < b < \infty \}$ and suppose $F$ is bounded in $S$. If $F$ has real zeros, $\{p_n\}$ in $S$, a necessary condition in order that $\sum_1^\infty d(p_n, \partial S) < \infty$ where $d(p_n, \partial S)$ denotes the distance from $p_n$ to the boundary of $S$. Conversely, given a real sequence $\{p_n\}, p_n \in S$, with $\sum_1^\infty d(p_n, \partial S) < \infty$, there exists a function which is analytic in $S$, bounded in $S$ and whose zeros in $S$ are precisely $\{p_n\}$.

**Proof.** The function defined by

$$s(z) = a + \frac{b - a}{\pi i} \log \left[ i \frac{1 + z}{1 - z} \right]$$

maps the unit disc $U = \{ z : |z| < 1 \}$ conformally onto $S$, and if $\alpha_n$ is the zero of $f(z) = F(s(z))$ corresponding to $p_n$, we must have

$$|\alpha_n|^2 = \frac{1 - \sin \pi \gamma_n}{1 + \sin \pi \gamma_n} \quad (\gamma_n = \frac{p_n - a}{b - a}).$$

Now, since $\sum_1^\infty (1 - |\alpha_n|)$ converges if and only if $\sum_1^\infty (1 - |\alpha_n|^2)$ converges, one readily verifies that $\sum_1^\infty (1 - |\alpha_n|) < \infty$ if and only if $\sum_1^\infty d(p_n, \partial S) < \infty$ and hence the lemma follows from Theorems 15.21 and 15.23 of [2, pp. 302–303].

**Proof of Theorem 2.** Let

$$F(s) = \int_0^\infty t^{p-1}(D_f - D_g)(t) \, dt \quad \left( \frac{1}{2} \leq \Re s \leq p_\infty \right).$$
Then $F(s)$ is the Mellin transform of $D_f - D_g$ which is analytic in $\frac{1}{2} \leq \Re s < p_\infty$, and since

$$|F(s)| \leq \left( \int_0^1 + \int_1^\infty \right) t^\Re s - 1 (D_f + D_g)(t) \, dt$$

$$\leq \int_0^1 t^{-1/2} \cdot 2 \cdot dt + \int_1^\infty t^p \cdot \cdot \cdot (D_f + D_g)(t) \, dt$$

$$= 4 + \| f \|_{p_\infty} + \| g \|_{p_\infty}^p$$

for $\frac{1}{2} \leq \Re s \leq p_\infty$, $F$ is bounded in the closed strip. Hence, by the lemma, $\sum_{p_n}^\infty (p_\infty - p_n) = \infty$ implies $F \equiv 0$ and hence $D_f - D_g = 0$ by the well-known inversion theorem for the Mellin transform (see [3, Theorem 9a, pp. 246–247]).

Conversely, given $1 \leq p_n \leq p_\infty < \infty$ such that $\sum_{p_n}^\infty (p_\infty - p_n) < \infty$, the lemma implies the existence of an analytic function $F(s) \neq 0$ for which $|F(s)| \leq M < \infty$, $-1 \leq \Re s \leq p_\infty$, and the zeros of $F$ are precisely the real numbers $p_n$, $n = 1, 2, \cdots$. Define

$$G(s) = e^{s^2} F(s), \quad -1 \leq \Re s \leq p_\infty.$$ 

Then with $s = \sigma + it$, $G(s)$ tends to zero uniformly in $-1 \leq \sigma \leq p_\infty$ as $|t| \to \infty$, and

$$\int_{-\infty}^{\infty} |G(\sigma + it)| \, dt \leq M \sqrt{\pi} e^{s^2}.$$ 

Hence, according to Theorem 19a of [3, p. 265],

$$G(s) = \int_{-\infty}^{\infty} e^{-sz} \phi(x) \, dx$$

where

$$\phi(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} G(s) e^{sz} \, ds$$

provided $-1 < \sigma < p_\infty$, $-\infty < x < \infty$; moreover (2) does not depend on $\sigma$. For $x > 0$, we put $\psi(x) = \phi(-\log x)$. Then

$$|\psi'(x)| \leq \frac{1}{2\pi} Me^{\sigma^2} \int_0^\infty \left( \sigma^2 + t^2 \right)^{1/2} e^{-t^2} x^{-(\sigma+1)} \, dt \leq Ax^{-(\sigma+1)}$$

where $A$ is a constant independent of $\sigma$, $-1 \leq \sigma \leq p_\infty$, and since $\psi$ is independent of $\sigma$, we must have

$$|\psi'(x)| \leq A \min_{-1/2 \leq \sigma \leq 1} \{ x^{-(\sigma+1)} \}$$

so that $\int_0^\infty |\psi'(x)| \, dx < \infty$ which shows that $\psi$ is of bounded variation on $(0, \infty)$. Let $\psi = \psi_1 - \psi_2$ where $\psi_i$ are nonnegative and nonincreasing.
Define $f$ and $g$ to be, respectively, the right continuous inverse of the functions $\varphi_1/c$ and $\varphi_2/c$ where $c = \sup |\varphi(x)| > 0$. Then $D_f = \varphi_1/c$, $D_g = \varphi_2/c$ and, if $1 \leq p \leq p_\infty$, 

$$\frac{1}{c} G(p) = \int_0^\infty t^{p-1} \left( \frac{\varphi_1}{c} - \frac{\varphi_2}{c} \right) (t) \, dt = \frac{1}{p} \left( \|f\|_p^p - \|g\|_p^p \right)$$

and since $G$ has zeros precisely at $p_n$, $n = 1, 2, \cdots$, we have $\|f\|_p = \|g\|_p$ if and only if $p \in \{p_n\}$.

REFERENCES


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