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ON L^p NORMS AND THE EQUIMEASURABILITY OF FUNCTIONS

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ABSTRACT. For measurable functions f and g, necessary and sufficient conditions are given for the equality of certain L^p norms of f and g to imply that f and g are equimeasurable.

Two Lebesgue measurable functions f and g defined on $I=[0,1]$ are said to be *equimeasurable* (or are rearrangements of each other) if they have the same distribution function, that is, if m denotes Lebesgue measure on I and

$$D_f(y) = m\{t:|f(t)| > y\} \qquad (y \geqq 0)$$

then $D_f=D_g$. It is very well known that the equimeasurability of f and g ensures that the L^p norms of f and g are equal for every p, $1\leqq p\leqq\infty$. Here, as usual, the L^p norm of f is defined by

$$\|f\|_p = \begin{cases} \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}, & 1 \leqq p < \infty, \\ \operatorname*{ess\,sup}_{0\leqq t\leqq 1} |f(t)|, & p = \infty. \end{cases}$$

In this paper we determine necessary and sufficient conditions in order that the equality of certain L^p norms of f and g will ensure that f and g are equimeasurable. More precisely we have the following results:

THEOREM 1. *Suppose f and g are essentially bounded functions on I and let $P=P(f,g)=\{p\geqq 1:\|f\|_p=\|g\|_p\}$. If P contains a sequence of distinct points $\{p_n\}$ with the property that $\sum_1^\infty (1/p_n)=\infty$ then f and g are equimeasurable, and in particular, $P=\{p:p\geqq 1\}$. Conversely, given a sequence $\{p_n\}$ with $p_n\geqq 1$ and $\sum_1^\infty (1/p_n)<\infty$, there exist bounded measurable functions f and g on I which are not equimeasurable, but $\|f\|_{p_n}=\|g\|_{p_n}$, $n=1,2,\cdots$.*

COROLLARY. *If P has a finite limit point, in particular if P is uncountable, then f and g are equimeasurable.*

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THEOREM 2. *Suppose f and g are measurable functions on I for which $\|f\|_p$ and $\|g\|_p$ are finite if and only if $1 \leqq p \leqq p_\infty < \infty$. If $P = P(f, g) = \{p : 1 \leqq p \leqq p_\infty, \|f\|_p = \|g\|_p\}$ contains a sequence of distinct points $\{p_n\}$ such that $\sum_1^\infty (p_\infty - p_n) = \infty$, then f and g are equimeasurable. Conversely, given p_∞, $1 < p_\infty < \infty$, and a sequence $\{p_n\}$, $1 \leqq p_n \leqq p_\infty$, such that $\sum_1^\infty (p_\infty - p_n) < \infty$, there exist measurable functions f and g defined on I with finite L^p norms if and only if $1 \leqq p \leqq p_\infty$ which are not equimeasurable but $\|f\|_p = \|g\|_p$ if and only if $p \in \{p_n\}$.*

The proof we give of Theorem 1 is an elementary application of the Hahn-Banach Theorem and the Theorem of Müntz. A second proof can be given which follows the line of our proof of Theorem 2, and which yields a slightly stronger conclusion in the "converse" part of Theorem 1, namely, the equality of the L^p norms of f and g if and only if $p \in \{p_n\}$. However, since Theorem 1 appears to be the most useful for applications, it seems desirable to sacrifice the additional strength in favour of a simple proof.

For the proof we require the following well-known result (see, for example, [**1**, Lemma 3.3.2, p. 182]):

$$\|f\|_p^p = p \int_0^\infty y^{p-1} D_f(y)\, dy \qquad (1 \leqq p < \infty). \tag{1}$$

PROOF OF THEOREM 1. (Sufficiency) By considering, if necessary, cf and cg where c is a constant, we may assume that both f and g are essentially bounded in absolute value by 1. Now if $C(I)$ denotes, as usual, the space of continuous functions on I with the L^∞ norm, then

$$L(h) = \int_0^1 h(t)(D_f(t) - D_g(t))\, dt \qquad (h \in C(I))$$

defines a bounded linear functional on $C(I)$ which by (1) vanishes on the functions $h_p(t) = t^p$, $p \in \{p_n - 1\}$. Now by Müntz' theorem (see, for example, [**2**, p. 305]) the functions h_p, $p > 0$, belong to the closure of the set $\{h_{p_n - 1} : n = 1, 2, \cdots\}$ and hence L vanishes on h_p, $p > 0$, by continuity of L. But then by the theorem of dominated convergence it follows that

$$L(h_0) = \int_0^1 \lim_{p \to 0+} t^p (D_f - D_g)(t)\, dt = \lim_{p \to 0+} L(h_p) = 0$$

and hence we must have $L = 0$, that is $D_f = D_g$.

(Necessity) Suppose that $\{p_n\}$ is given with $\sum_1^\infty (1/p_n) < \infty$. Then, again by Müntz' theorem, the set of functions $S = \{h_p : p = 0 \text{ or } p = p_n,\ n = 1, 2, \cdots\}$ is not dense in $C(I)$, and hence there is a bounded linear functional L, $\|L\| \leqq 1$, on $C(I)$ which vanishes on S but $L(h_{p_0}) \neq 0$ for some $p_0 > 1$.

Now, according to the representation theorem for bounded linear functionals on $C(I)$, there exist nonnegative, nonincreasing functions α, β, defined on I such that

$$L(h) = \int_0^1 h(t)\, d(\alpha - \beta)(t) \qquad (h \in C(I))$$

and $\alpha(1)=\beta(1)=0$. Moreover, we have $\alpha \leqq 1$ and $\beta \leqq 1$ since $\alpha(0)+\beta(0)=$ Total Variation of $(\alpha-\beta)=\|L\| \leqq 1$. Hence, if we define f and g to be, respectively, the right continuous inverse of α and β, then $D_f=\alpha$, $D_g=\beta$ and if $p \geqq 1$ integration by parts yields

$$L(h_p) = \int_0^1 t^p\, d(D_f - D_g) = -p\int_0^1 t^{p-1}(D_f - D_g)\, dt = \|g\|_p^p - \|f\|_p^p$$

so that $\|f\|_p=\|g\|_p$ for $p \in \{p_n : n=1, 2, \cdots\}$ but $\|f\|_{p_0} \neq \|g\|_{p_0}$ and in particular, $D_f \neq D_g$.

The proof of Theorem 2 requires the following lemma:

LEMMA. *Let $F(s)$ be an analytic function in the strip $S=\{s: -\infty<a<\operatorname{Re} s<b<\infty\}$ and suppose F is bounded in $\bar{S}$. If F has real zeros, $\{p_n\}$ in S, a necessary condition in order that $F \not\equiv 0$ is that $\sum_1^\infty d(P_n, \partial S)<\infty$ where $d(p_n, \partial S)$ denotes the distance from p_n to the boundary of S. Conversely, given a real sequence $\{p_n\}$, $p_n \in S$, with $\sum_1^\infty d(p_n, \partial S)<\infty$, there exists a function which is analytic in S, bounded in $\bar{S}$ and whose zeros in S are precisely $\{p_n\}$.*

PROOF. The function defined by

$$s(z) = a + \frac{b-a}{\pi i} \log\left[i\,\frac{1+z}{1-z}\right]$$

maps the unit disc $U=\{z : |z|<1\}$ conformally onto S, and if α_n is the zero of $f(z)=F(s(z))$ corresponding to p_n, we must have

$$|\alpha_n|^2 = \frac{1 - \sin \pi\gamma_n}{1 + \sin \pi\gamma_n} \qquad \left(\gamma_n = \frac{p_n - a}{b - a}\right).$$

Now, since $\sum_1^\infty (1-|\alpha_n|)$ converges if and only if $\sum_1^\infty (1-|\alpha_n|^2)$ converges, one readily verifies that $\sum_1^\infty (1-|\alpha_n|)<\infty$ if and only if $\sum_1^\infty d(p_n, \partial S)<\infty$ and hence the lemma follows from Theorems 15.21 and 15.23 of [**2**, pp. 302–303].

PROOF OF THEOREM 2. Let

$$F(s) = \int_0^\infty t^{s-1}(D_f - D_g)(t)\, dt \qquad (\tfrac{1}{2} \leqq \operatorname{Re} s \leqq p_\infty).$$

Then $F(s)$ is the Mellin transform of $D_f - D_g$ which is analytic in $\frac{1}{2} < \operatorname{Re} s < p_\infty$, and since

$$\begin{aligned} |F(s)| &\leqq \left(\int_0^1 + \int_1^\infty\right) t^{\operatorname{Re} s - 1}(D_f + D_g)(t)\,dt \\ &\leqq \int_0^1 t^{-1/2} \cdot 2 \cdot dt + \int_0^\infty t^{p_\infty - 1}(D_f + D_g)(t)\,dt \\ &\leqq 4 + \|f\|_{p_\infty}^{p_\infty} + \|g\|_{p_\infty}^{p_\infty} \end{aligned}$$

for $\frac{1}{2} \leqq \operatorname{Re} s \leqq p_\infty$, F is bounded in the closed strip. Hence, by the lemma, $\sum_1^\infty (p_\infty - p_n) = \infty$ implies $F \equiv 0$ and hence $D_f - D_g = 0$ by the well-known inversion theorem for the Mellin transform (see [3, Theorem 9a, pp. 246–247]).

Conversely, given $1 \leqq p_n \leqq p_\infty < \infty$ such that $\sum_1^\infty (p_\infty - p_n) < \infty$, the lemma implies the existence of an analytic function $F(s) \not\equiv 0$ for which $|F(s)| \leqq M < \infty$, $-1 \leqq \operatorname{Re} s \leqq p_\infty$, and the zeros of F are precisely the real numbers p_n, $n = 1, 2, \cdots$. Define

$$G(s) = e^{s^2} F(s), \qquad -1 \leqq \operatorname{Re} s \leqq p_\infty.$$

Then with $s = \sigma + it$, $G(s)$ tends to zero uniformly in $-1 \leqq \sigma \leqq p_\infty$ as $|t| \to \infty$, and

$$\int_{-\infty}^\infty |G(\sigma + it)|\,dt \leqq M\sqrt{\pi}\, e^{\sigma^2}.$$

Hence, according to Theorem 19a of [3, p. 265],

$$G(s) = \int_{-\infty}^\infty e^{-sx} \phi(x)\,dx$$

where

$$\phi(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} G(s) e^{xs}\,ds \tag{2}$$

provided $-1 < \sigma < p_\infty$, $-\infty < x < \infty$; moreover (2) does not depend on σ. For $x > 0$, we put $\psi(x) = \phi(-\log x)$. Then

$$|\psi'(x)| \leqq \frac{1}{2\pi} M e^{\sigma^2} \int_0^\infty (\sigma^2 + t^2)^{1/2} e^{-t^2} x^{-(\sigma+1)}\,dt \leqq A x^{-(\sigma+1)}$$

where A is a constant independent of σ, $-1 \leqq \sigma \leqq p_\infty$, and since ψ is independent of σ, we must have

$$|\psi'(x)| \leqq A \min_{-1/2 \leqq \sigma \leqq 1} \{x^{-(\sigma+1)}\}$$

so that $\int_0^\infty |\psi'(x)|\,dx < \infty$ which shows that ψ is of bounded variation on $(0, \infty)$. Let $\psi = \psi_1 - \psi_2$ where ψ_i are nonnegative and nonincreasing.

Define f and g to be, respectively, the right continuous inverse of the functions ψ_1/c and ψ_2/c where $c=\sup|\psi(x)|>0$. Then $D_f=\psi_1/c$, $D_g=\psi_2/c$ and, if $1\leqq p\leqq p_\infty$,

$$\frac{1}{c}G(p)=\int_0^\infty t^{p-1}\left(\frac{\psi_1}{c}-\frac{\psi_2}{c}\right)(t)\,dt=\frac{1}{p}(\|f\|_p^p-\|g\|_p^p)$$

and since G has zeros precisely at p_n, $n=1, 2, \cdots$, we have $\|f\|_p=\|g\|_p$ if and only if $p\in\{p_n\}$.

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