ON VALUATION RINGS THAT CONTAIN ZERO DIVISORS

JAMES A. HUCKABA

Abstract. Let R be a commutative ring with identity. A new proof is given of the theorem due to Samuel and Griffin which states that R is integrally closed in its total quotient ring if and only if R is the intersection of B-valuation rings. We then prove the main result of the paper: If K is a π-regular ring, then K admits only Prüfer rings as valuation rings.

1. A ring means a commutative ring with identity. If (G, +) is a totally ordered abelian group, then a function v from a ring T into \( G \cup \{\infty\} \) is an evaluation with value group G, if for all \( x, y \in T \):

1. \( v(xy) = v(x) + v(y) \);
2. \( v(x + y) \geq \min\{v(x), v(y)\} \);
3. \( v(1) = 0 \) and \( v(0) = \infty \).

If v maps T onto \( G \cup \{\infty\} \), then v is called a valuation. For any value group G, let \( G^+ = \{\alpha \in G : \alpha \geq 0\} \cup \{\infty\} \). Let R be a subring of the ring T and suppose that v is an evaluation with value group G. We consider four conditions on R.

1. \( R = v^{-1}(G^+) \).
2. v is a valuation and \( R = v^{-1}(G^+) \).
3. If \( P(X_1, \ldots, X_r) \) is a dominated polynomial over R, then \( P(s_1, \ldots, s_r) \neq 0 \) for every set \( \{s_i\}_{i=1}^r \subset T - R \). A dominated polynomial over R is a polynomial of the form \( P(X_1, \ldots, X_r) = X_1^{n(1)} \cdots X_r^{n(r)} + \sum_\delta a_\delta X_1^{\delta(1)} \cdots X_r^{\delta(r)} \), with \( a_\delta \in R \) and \( (n(1), \ldots, n(r)) > (\delta(1), \ldots, \delta(r)) \) where the order is given by the ordered product of r copies of the natural numbers.
4. R contains a prime ideal P such that if B is a ring between R and T, and if Q is a prime ideal of B contracting to P, then \( R = B \).

If we assume that T is a field, then the above definitions are equivalent and correspond to the classical definition of a valuation domain. In [13]
Samuel gives several generalizations of valuation theory to arbitrary commutative rings. It was this paper that introduced $(V_3)$ and $(V_4)$. Manis [10] proved the equivalence of $(V_2)$ and $(V_4)$, and at the present time it is commonly accepted that the best generalization of a valuation domain is a ring satisfying these conditions. Such rings are, therefore, called valuation rings. The concept that we call an evaluation was introduced by Bourbaki in [2] as his definition of a valuation. Hence, we call rings satisfying $(V_1)$ B-valuation rings.

Krull proved in 1932 [7] that the integral closure of an integral domain $D$ is the intersection of valuation domains between $D$ and its quotient field. Samuel generalized this in 1957 when he showed that the integral closure of an arbitrary commutative ring $R$ in an overring $T$ is the intersection of rings of type $(V_3)$. Then in 1970 Griffin [5] proved the equivalence of $(V_3)$ and $(V_4)$, thus showing:

**Theorem 1.** A ring $R$ is integrally closed in an overring $T$ if and only if $R$ is the intersection of B-valuation rings between $R$ and $T$.

The most important case of Theorem 1 is when $T$ is assumed to be the total quotient ring of $R$. For this situation we give, in §2, a new short proof of Theorem 1. Our method is completely different than that of Samuel and Griffin, and avoids the use of $(V_3)$, the most complicated of the $(V_t)$ axioms.

The methods of §2 are applicable to another important problem that arises when studying valuations on commutative rings—the problem of determining when a valuation ring is a Prüfer ring. The main part of this paper, §3, is devoted to this problem. We show that if $K$ is a $\pi$-regular ring, then every valuation subring of $K$ is Prüfer (Theorem 6).

2. Let $R$ be a ring, let $T(R)$ be its total quotient ring, and let $Q(R)$ denote its complete ring of quotients. For the definition and pertinent facts about the complete ring of quotients see Lambek’s book [8]. In this section we give a short proof of Theorem 1. We first need a result that is an easy generalization of one part of [6, Proposition 9]. The proof is similar to Griffin’s proof and will not be repeated.

**Proposition 2.** If $K$ is a von Neumann regular ring containing the ring $R$, then $R$ is integrally closed in $K$ if and only if $R$ is equal to an intersection of valuation rings of $K$ that contain $R$.

Note that in Proposition 2, $K$ is not necessarily the total quotient ring of $R$.

**Proof of Theorem 1.** We first assume that $R$ is a reduced ring (i.e., $R$ contains no nonzero nilpotent elements). Denote the integral closure of $R$
in $T(R)$ (resp., $Q(R)$) by $\bar{R}$ (resp., $\bar{R}$). By [8, p. 42], $Q(R)$ is a von Neumann regular ring. Proposition 2 implies that $\bar{R} = \bigcap V_{\alpha}$, where the $V_{\alpha}$ are valuation rings of $Q(R)$. Thus, since $\bar{R} \subseteq T(R) \subseteq Q(R)$,

$$(1) \quad \bar{R} = \bar{R} \cap T(R) = (\bigcap V_{\alpha}) \cap T(R) = \bigcap V_{\alpha},$$

where $V'_{\alpha} = V_{\alpha} \cap T(R)$. If $v_{\alpha}$ is the valuation on $Q(R)$ corresponding to $V_{\alpha}$, if $G_{\alpha}$ is the value group of $v_{\alpha}$, if $w_{\alpha}$ is the restriction of $v_{\alpha}$ to $T(R)$, and if $H_{\alpha}$ is the subgroup of $G_{\alpha}$ generated by the range of $w_{\alpha}$, then it is clear that $w_{\alpha}$ is an evaluation with value group $H_{\alpha}$ and that $w_{\alpha}^{-1}(H_{\alpha}) = V'_{\alpha}$.

We now drop the assumption that $R$ is reduced. Let $N$ be the nilradical of $T(R)$, then $N_0 = N \cap R$ is the nilradical of $R$. Canonically embed $R/N_0$ into $T(R)/N$ and embed $T(R)/N$ into $T(R/N_0)$ via the mapping $x/y + N \mapsto (x + N_0)/(y + N_0)$. Hence we may assume that

$$(2) \quad R/N_0 \subseteq T(R)/N \subseteq T(R/N_0) \subseteq Q(R/N_0).$$

It follows from the first paragraph of this proof that $(R/N_0)^{\sim} = \bigcap V_{\alpha}$, where each $V_{\alpha}$ is a valuation ring with respect to $Q(R/N_0)$. Again we assume that $v_\alpha$ and $G_\alpha$ are the valuation and valuation group associated with $V_\alpha$. Let $v'_\alpha$ be the restriction of $v_\alpha$ to $T(R)/N$, and define $w_\alpha : T(R) \to G_\alpha \cup \{\infty\}$ such that $w_\alpha(x) = v'_\alpha(x + N)$. It follows that $w_\alpha$ is an evaluation on $T(R)$ such that $w_\alpha$ is infinite on $N$. If $H_\alpha$ is the subgroup of $G_\alpha$ generated by the range of $w_\alpha$, then $W_\alpha = w_\alpha^{-1}(H^+_\alpha)$ is a $B$-valuation ring of $T(R)$.

It remains to show that $\bar{R} = \bigcap W_{\alpha}$. Each $W_{\alpha} \supseteq R$ and is integrally closed [13, Theorem 1], hence $\bar{R} \subseteq \bigcap W_{\alpha}$. Assume that $x \in \bigcap W_{\alpha}$. For each $\alpha$, $w_\alpha(x) = v'_\alpha(x + N) \geq 0$, which implies that $x + N$ is in the integral closure of $R/N_0$ in $Q(R/N_0)$. Write $(x + N)^{t_1} + (r_1 + N_0)(x + N)^{t_2} + \cdots + (r_t + N_0) = 0 + N$. We have $x^{t_1} + r_1 x^{t_2} + \cdots + r_t \in N$, so there exists a positive integer $s$ such that $(x^{t_1} + r_1 x^{t_2} + \cdots + r_t)^s = 0$. This last equation shows that $x$ is integral over $R$. Therefore, $\bar{R} = \bigcap W_{\alpha}$.

3. Let $K$ denote an arbitrary total quotient ring. For a valuation ring $V$ of $K$ we have $V \subseteq T(V) \subseteq K$. It is easy to see that $V$ is also a valuation ring of $T(V)$. A ring $R$ is a Prüfer ring in case the rings between $R$ and $T(R)$ are integrally closed. In the classical case every valuation domain is a Prüfer domain. This is not true in rings with zero divisors, see [1] and [3]. The purpose of this section is to give some sufficient conditions on the ring $K$ that imply that every valuation ring of $K$ is Prüfer. These conditions are given in Lemma 3, Theorem 6, and Corollary 7.

A ring $R$ has few zero divisors if it has only finitely many maximal prime ideals of zero. $R$ is additively regular if, for each $z \in T(R)$, there is a $u \in R$ such that $z + u$ is a regular element of $T(R)$ (see [4]). The class of additively regular rings contains the set of rings with few zero divisors which, in
A ring $R$ is $\pi$-regular in case, for each $r \in R$, there exists an $a \in R$ and a positive integer $n$ such that $r^n = (r^n)^2 a$. Note that if $n=1$ we merely have the definition of a commutative von Neumann regular ring. Recall that a subdirect sum of rings $\{R_\alpha\}_{\alpha \in A}$ is a subring $S$ of the complete direct sum $\bigoplus \sum R_\alpha = \{f : \mathcal{A} \to \bigcup R_\alpha | f(\alpha) \in R_\alpha, \text{ for each } \alpha \in \mathcal{A}\}$ with the property that for each $r_\alpha \in R_\alpha$, there exists $f \in S$ so that $f(\alpha) = r_\alpha$ [11]. We need the following two types of quotient rings. Let $N$ be a multiplicatively closed subset of $R$. Define

\[ R_{(N)} = \{a/b : a, b \in R, b \text{ is a regular element of } N\}; \]
\[ R_{[N]} = \{z \in T(R) : zs \in R, \text{ for some } s \in N\}. \]

The ring $R_{(N)}$ is called the large quotient ring of $R$ with respect to $N$. If $P$ is a prime ideal of $R$ we write $R_{(P)}$ and $R_{[P]}$ in place of $R_{(R-P)}$ and $R_{[R-P]}$, respectively. A quasi-valuation ring is a ring $R$ with the property that for each regular element $x$ in $T(R)$, $x$ or $x^{-1}$ is in $R$.

**Lemma 3.** Let $R$ be an additively regular ring.

(a) $R$ is a valuation ring if and only if $R$ is a quasi-valuation ring.

(b) If $P$ is a prime ideal of $R$, then $R_{[P]} = R_{(P)}$.

(c) If $V$ is a valuation ring between $R$ and $T(R)$, then $V$ is a Prüfer ring.

**Proof.** (a) and (b) are in [4], and are easy generalizations of Lemmas 2 and 4 of [6].

(c) It is obvious that $V$ is additively regular. Let $M$ be a maximal ideal of $V$. Then (a) and (b) imply that $V_{[M]} = V_{(M)}$ is a valuation ring. By Theorem 13 of [6], $V$ is a Prüfer ring.

**Lemma 4.** (a) If $K$ is a $\pi$-regular ring and if $V$ is a valuation ring of $K$; then $T(V)$ is a $\pi$-regular ring, or $V = T(V)$.

(b) If $K$ is a von Neumann regular ring and if $V$ is a valuation ring of $K$; then $T(V)$ is a von Neumann regular ring, or $V = T(V)$.

**Proof.** Recalling that $(V_2)$ is equivalent to $(V_4)$, we let $P$ (resp., $v$) be the prime (resp., valuation) associated with the valuation ring $V$ as in $(V_4)$ (resp., $(V_2)$). The ideal $P_\infty = \{x \in T(V) : v(x) = \infty\}$ is a prime ideal in both $V$ and $T(V)$.

(a) If $V$ contains no regular nonunits, then $V$ is a total quotient ring and is therefore equal to $T(V)$. So we assume the existence of at least one regular nonunit $b$ of $V$. We must show that if $z \in T(V)$, then there is an $a \in T(V)$ and a positive integer $n$ such that $z^na = z^n$.

Case 1. If $z \notin V$, then there is an $a \in K$ such that $z^na = z^n$ for some $n$. But, $nv(z) = -v(a) < 0$ implies that $a \in V \subseteq T(V)$. 

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Case 2. Assume that \( z \in V - P \). Then \( v(z/b) < 0 \). So by Case 1, \( (z/b)^2 a = (z/b)^n \), and thus \( z^{2n}(a/b^n) = z^n \) where \( a/b^n \in T(V) \).

Case 3. Let \( z \in P - P_\infty \). Since \( V \) is a valuation ring of \( T(V) \), we can find \( y \in T(V) - V \) such that \( zy \in V - P \). Say that \( y = c/d \), where \( d \) is a regular element of \( R \). It follows that \( z/d \notin P \). Use Case 1 if \( z/d \in T(V) - V \), and use Case 2 if \( z/d \in V - P \).

Case 4. Let \( z \in P_\infty \), then there exist \( a \) and \( n \) as in Case 1 so that \( z^{2n} a = z^n \). Also, \( z^{2n}(a^2 z^n) = z^n \), and clearly \( a^2 z^n \in P_\infty \subseteq T(V) \). The proof of (b) follows from the proof of (a) by letting \( n = 1 \).

Proposition 5. If \( R \) is a ring whose total quotient ring \( T(R) \) is \( \pi \)-regular, then \( R \) is additively regular. In particular, every ring with von Neumann regular total quotient ring is additively regular.

Proof. We first prove the result when \( T(R) \) is assumed to be a von Neumann regular ring. It is well known that a von Neumann regular ring is a subdirect sum of a set of fields—say that \( T(R) \) is a subdirect sum of the family of fields \( \{F_a\}_{a \in \mathcal{A}} \). If \( z \in T(R) \), write \( z = ab \) where \( a, b \in R \) and \( b \) is regular. Considering \( a \) as an element of \( T(R) \), there is an element \( c \in T(R) \) such that \( a^2 c = a \). Write \( c = c_0/c_1 \) where \( c_i \in R \) and \( c_1 \) is regular.

Define \( u = (c_1 - ac_0)b \) and let \( t = z + u \). Our goal is to prove that \( t \) is regular. This is equivalent to proving \( t(\alpha) \neq 0 \), for all \( \alpha \in \mathcal{A} \). (Here we are treating \( T(R) \) as a subdirect sum of \( \{F_a\}_{a \in \mathcal{A}} \).) Fix \( \alpha \in \mathcal{A} \), then \( t(\alpha) = a(\alpha)b(\alpha) + [c_1(\alpha) - a(\alpha)c_0(\alpha)]b(\alpha) \). If \( a(\alpha) = 0 \), then \( t(\alpha) = c_1(\alpha)b(\alpha) \neq 0 \). On the other hand, assume that \( a(\alpha) \neq 0 \). Then \( c_1(\alpha) - a(\alpha)c_0(\alpha) = 0 \); for \( a^2 c = a \) implies that \( a(\alpha)[a(\alpha)c(\alpha) - 1] = 0 \), and hence

\[
(a(\alpha)c(\alpha) - 1 = a(\alpha)[c_0(\alpha)/c_1(\alpha)] - 1 = 0.
\]

Thus \( t(\alpha) = a(\alpha)b(\alpha) \neq 0 \).

Now assume that \( T(R) \) is \( \pi \)-regular. Let \( \{P_\delta\} \) be the set of prime ideals of \( T(R) \) and let \( N = \bigcap P_\delta \) be the nilradical of \( N \). By relation (2), \( R/(R \cap N) \subseteq T(R)/N \subseteq T(R)/(R \cap N) \) and since \( T(R)/N \) is a von Neumann regular ring \([12, \text{Proposition 1}]\), we must have \( T(R)/N = T(R)/(R \cap N) \). By the first part of the proof, \( R/(R \cap N) \) is additively regular. Thus, if the coset \( z + N \) is in \( T(R)/N \), there exists a \( u \in R \) such that \( (z + N) + (u + N) = b + N \) is regular in \( T(R)/N \). Clearly \( z + u = b + n \), where \( n \in N \). The regular element \( b + N \notin P_\delta/N \) for all \( \delta \), and thus \( b \notin P_\delta \) for all \( \delta \). It follows that \( b + n \) is a regular element of \( T(R) \). This completes the proof.

Theorem 6. If \( K \) is a \( \pi \)-regular ring, then every valuation ring of \( K \) is a Prüfer ring.

Proof. Let \( V \) be a valuation ring of \( K \). By Lemma 4, \( V = T(V) \) or
$T(V)$ is $\pi$-regular. In the first case $V$ is obviously Prüfer. In the second case invoke Lemma 3 and Proposition 5.

Returning to the situation in §2, we have

**Corollary 7.** If $R$ is a reduced ring, then every valuation ring of $\mathcal{Q}(R)$ is a Prüfer ring.

4. In this last section we consider rings with the following property:

(I) $R$ is integrally closed in $T(R)$ if and only if $R$ is the intersection of valuation rings between $R$ and $T(R)$.

If $R$ is a domain, then (I) is Krull’s theorem. There are two results in the literature of a ring $R$ satisfying (I), when $R$ is assumed to have zero divisors. Griffin [6] and Larsen [9] proved independently that (I) holds for rings with few zero divisors. Also appearing in [6] is the result that if $T(R)$ is a von Neumann regular ring, then (I) holds for $R$. It is quite easy to see, using Griffin’s methods, that additively regular rings satisfy (I). We give another class of rings satisfying (I).

**Proposition 8.** Let $R$ be a ring such that $\mathcal{Q}(R)$ is a von Neumann regular ring and such that every unit of $\mathcal{Q}(R)$ is also a unit of $T(R)$. Then $R$ satisfies property (I).

**Proof.** Assume that $R$ is integrally closed in $T(R)$ and that $R'$ is the integral closure of $R$ in $\mathcal{Q}(R)$. By Proposition 2, $R'=\bigcap V_a$, where each $V_a$ is a valuation ring of $\mathcal{Q}(R)$. From relation (1) $R=\bigcap V'_a$, where $V'_a=V_a\cap T(R)$. To complete the proof we need show that each $V'_a$ is a valuation ring. To this end, we prove that each $V'_a$ is additively regular (for it is already a quasi-valuation ring). Let $z\in T(R)=T(V'_a)$. By Lemma 4 and Proposition 5, $V_a$ is additively regular. Choose $u\in V_a$ so that $z+u=r$ is regular in $T(V_a)$. Thus, $r$ is regular in $\mathcal{Q}(R)$, so $r\in T(R)$, and hence $r-z=u\in V'_a$. Therefore $V'_a$ is additively regular.

Rings satisfying the hypothesis of Proposition 8 are easy to construct. For example, for each natural number $i$, let $F_i$ be the prime field of characteristic 2, let $K$ be the complete direct sum of $\{F_i\}_{i=1}^\infty$, and let $R$ be the subring of $K$ generated by the discrete direct sum of $\{F_i\}_{i=1}^\infty$ and the identity element of $K$. Then $R=T(R)$, $K=\mathcal{Q}(R)$, and $\mathcal{Q}(R)\neq T(R)$ since $(1,0,1,0,\cdots)\notin T(R)$. Clearly, every unit of $\mathcal{Q}(R)$ is a unit of $T(R)$.

**Bibliography**


Department of Mathematics, University of Missouri, Columbia, Missouri 65201