EQUATIONS WHICH CHARACTERIZE INNER PRODUCT SPACES

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Abstract. It is shown that if $N$ is a normed linear space and there is a point $y$ of norm 1 such that an inequality of the type $a^2\|x\|^2 \leq \lim_{u \to 0} G(\{\|b_i u x + c_i y\|\}_{i=1}^n) \leq b^2\|x\|^2$ holds for all $x$ in $N$ (where $0 < a \leq b$, the $c_i$'s are nonzero and $G$ and $\|\cdot\|$ satisfy a certain twice-differentiability condition), then $N$ is isomorphic to an inner product space and $\inf\|T\| \cdot \|T^{-1}\| \leq b/a$, where the infimum is taken over all linear homeomorphisms $T$ between $N$ and an inner product space. In the event that $a = b = 1$, the inequality reduces to an equation which characterizes inner product spaces. An example shows that these results do not follow without the twice-differentiability condition on $G$.

S. O. Carlsson has proved [1] that a normed linear space $N$ is an inner product space if an equation of the type

\[ \sum_{i=0}^{n} a_i \|b_i x + c_i y\|^2 = 0 \]

(where the numbers $a_i$ are nonzero and the couples $(b_i, c_i)$ are pairwise linearly independent) holds for all $x$ and $y$ in $N$. An example of this type of equation is the Jordan-von Neumann condition for inner product spaces [2]:

\[ \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \]

for all $x$ and $y$ in $N$. A reformulation of Carlsson’s condition is the following:

\[ \|x\|^2 = \sum_{i=1}^{n} a_i \|b_i x + c_i y\|^2 \]

(where the coefficients are different from those in (1) and the numbers $c_i$ are nonzero) for all $x$ and $y$ in $N$. Now, if we define a function $G_0$ by

\[ G_0(t_1, t_2, \cdots, t_n) = \sum_{i=1}^{n} a_i t_i^2 \]

then equation (3) says that, for all $x$ and $y$ in $N$,

\[ \|x\|^2 = G_0(\{\|b_i x + c_i y\|\}_{i=1}^n). \]

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In this paper, Carlsson's condition is improved upon in several ways simultaneously: the specific function $G_0$ is replaced by a function $G$ which only need be twice-differentiable at a certain point, the requirement that (4) hold for all $y$ is replaced by the condition that it hold for a single $y$ (on the unit sphere for convenience) where the norm is twice-differentiable, the requirement that (4) hold for all $x$ is replaced by a weaker limit condition, and the equality in (4) is replaced by inequalities which give information on the nearness of $N$ to an inner product space in case $N$ is isomorphic but not isometric to an inner product space. The main result of this paper is: If there is a point $y$ on the unit sphere and some function $G$ such that an inequality of the type

$$a^2 \|x\|^2 \leq \lim_{u \to 0} \frac{G(\|b_1ux + c_1y\|)}{u^2} \leq b^2 \|x\|^2$$

holds for all $x$ in $N$ (where $0 < a \leq b$, the numbers $c_i$ are nonzero, and $G$ and the norm satisfy a certain twice-differentiability condition), then $N$ is isomorphic to an inner product space. Furthermore, letting $K(N) = \inf\{\|T\| \cdot \|T^{-1}\| : T$ is an isomorphism from $N$ onto an inner product space$, it follows that $K(N) \leq b/a$. (This means that an inner product $(\cdot, \cdot)$ exists such that $a^2 \|x\|^2 \leq (x, x) \leq b^2 \|x\|^2$ for all $x$ in $N$.) An example shows that these results do not follow without the twice-differentiability condition on $G$. The twice-differentiability condition on the norm can be removed if we require that (5) hold for all $x$ and $y$ in $N$.

Say that a function $G$ from a subset $A$ of a normed linear space $N_1$ into a normed linear space $N_2$ is twice-differentiable at the point $x$ of $A$ if $x$ is a limit point of $A$ and there exist a bounded linear mapping $T_1$ from $N_1$ into $N_2$ and a bounded bilinear mapping $T_2$ from $N_1 \times N_1$ into $N_2$ such that

$$\lim_{\varepsilon \to 0, x + \varepsilon \in A} \frac{G(x + \varepsilon) - G(x) - T_1(\varepsilon) - \frac{1}{2}T_2(\varepsilon, \varepsilon)}{\|\varepsilon\|^2} = 0.$$ 

This definition does not require that $G$ be differentiable at points of $A$ other than $x$. The necessary lemmas on this subject are stated at the end.

Throughout this paper, $N$ denotes a normed linear space. An $n$-tuple $(t_1, t_2, \cdots, t_n)$ is denoted $\{t_i\}$. If $t_0, t_1, \cdots, t_n$ are numbers or points, $\{t_i\}$ denotes $(t_1, t_2, \cdots, t_n)$, not $(t_0, t_1, \cdots, t_n)$.

**Theorem 1.** Suppose that there exist a function $G$ from $N$ into $R$ which is twice-differentiable at 0 and numbers $a$ and $b$, $0 < a \leq b$, such that for all $x$ in $N$, $a^2 \|x\|^2 \leq \lim_{u \to 0} G(ux)/u^2 \leq b^2 \|x\|^2$. Then $K(N) \leq b/a$.

**Proof.** Let $T_1$ and $T_2$ be derivatives of $G$ at 0, as in the above definition.
Then
\[ \lim_{x \to 0} \frac{G(x) - G(0) - T_1(x) - \frac{1}{2}T_2(x, x)}{\|x\|^2} = 0. \]

This implies that \( G \) is continuous at 0, and since \( \lim_{u \to 0} G(ux)/u^2 \) exists, \( G(0)=0 \). For all \( x \) in \( N \), \( 0 = \lim_{u \to 0} (G(ux) - uT_1(x) - \frac{1}{2}u^2T_2(x, x))/u^2 = (\lim_{u \to 0} G(ux)/u^2) - \frac{1}{2}T_2(x, x) - (\lim_{u \to 0} T_1(x))/u \). Therefore, \( T_1 = 0 \) and \( \lim_{u \to 0} G(ux)/u^2 = \frac{1}{2}T_2(x, x) \). For all \( x \) and \( y \) in \( N \), define \( ((x, y)) = (T_2(x, y) + T_2(y, x))/4 \). Then \( ((\cdot, \cdot)) \) is an inner product for \( N \) satisfying \( a^2\|x\|^2 \leq ((x, x)) \leq b^2\|x\|^2 \) for all \( x \) in \( N \). This implies \( K(N) \leq b/a \).

**Corollary 1.1.** If the square of the norm is twice-differentiable at 0, then \( N \) is an inner product space.

**Theorem 2.** Suppose that there exist a point \( y \) of norm 1 where \( \|\cdot\| \) is twice-differentiable, numbers \( b_i \) and \( c_i \), \( i=1, \ldots, n \), a function \( G \) from a subset of \( \mathbb{R}^n \) into \( \mathbb{R} \) twice-differentiable at \( \{|c_i|\} \), and numbers \( a \) and \( b \), \( 0 < a \leq b \), such that for every \( x \) in \( N \), \( a^2\|x\|^2 \leq \lim_{u \to 0} G(\|b_ix+c_iy\|)/u^2 \leq b^2\|x\|^2 \). Then \( K(N) \leq b/a \).

**Proof.** Let \( \mathcal{N} = N_1 \times N_2 \times \cdots \times N_n \), where each \( N_i = N \). Then define functions \( E: \mathbb{R} \to \mathcal{N} \) and \( F: \mathcal{N} \to \mathbb{R}^n \) by \( E(u) = \{b_iux+c_iy\} \) and \( F(p_i) = \|p_i\| \). Then \( E \) is twice-differentiable at 0, \( F \) is twice-differentiable at \( E(0) \), and \( G \) is twice-differentiable at \( F \circ E(0) \). By two applications of Lemma 1 on twice-differentiable functions, \( G \circ F \circ E \) is twice-differentiable at 0. By the hypothesis \( a^2\|x\|^2 \leq \lim_{u \to 0} G \circ F \circ E(ux)/u^2 \leq b^2\|x\|^2 \) for all \( x \) in \( N \). Theorem 1 then asserts that \( K(N) \leq b/a \).

One immediate consequence of Theorem 2 is the following improvement on the Jordan-von Neumann condition:

**Corollary 2.1.** If there exist a point \( y \) of norm 1 in \( N \) where \( \|\cdot\| \) is twice-differentiable and numbers \( a \) and \( b \), \( 0 < a \leq b \), such that for all \( x \) in \( N \),

\[ a^2\|x\|^2 \leq \lim_{u \to 0} \frac{\|ux + y\|^2 + \|ux - y\|^2 - 2\|y\|^2}{2u^2} \leq b^2\|x\|^2, \]

then \( K(N) \leq b/a \).

**Proof.** It is just a matter of checking that the function \( G(t_1, t_2, t_3) = (t_1^2 + t_2^2 - 2t_3^2)/2 \) is twice-differentiable at \( (1, 1, 1) \).

If \( N \) is 2-dimensional, then Lemma 2 on twice-differentiable functions may be used to eliminate from the hypothesis of Corollary 2.1 the condition that there exist a point \( y \) of norm 1 where \( \|\cdot\| \) is twice-differentiable.

**Corollary 2.2.** Suppose that there exist a point \( y \) of norm 1 where \( \|\cdot\| \) is twice-differentiable, pairwise linearly independent points in the plane.
(b_i, c_i), i=0, 1, \ldots, n, a function G twice-differentiable at
\{b_0 c_i - c_0 b_i \}/(b_0^2 + c_0^2)^{1/2},
and numbers a and b, 0 \leq a \leq b, such that for all p and q in N, \[a^2 \|b_0 p + c_0 q\|^2 \leq G(\|b_0 p + c_0 q\|) \leq a^2 \|b_0 p + c_0 q\|^2.\] Then \(K(N) \leq b/a.\)

Proof. Define \(b'_i = (b_0 b_i + c_0 c_i)/(b_0^2 + c_0^2)^{1/2}, c'_i = (b_0 c_i - c_0 b_i)/(b_0^2 + c_0^2)^{1/2}, i=0, 1, \ldots, n.\) Then the \(c'_i\) are nonzero and \((b'_0, c'_0) = (1, 0).\) Suppose \(x\) is in \(N.\) Let \(p = (c_0 x - b_0 y)/(b_0^2 + c_0^2)^{1/2}\) and \(q = (c_0 x + b_0 y)/(b_0^2 + c_0^2)^{1/2}.\) Then
\[a^2 \|x\|^2 = a^2 \|b_0 p + c_0 q\|^2 \leq G(\|b_0 p + c_0 q\|) \leq G(\|b'_i x + c'_i y\|) \leq b^2 \|b_0 p + c_0 q\|^2 = b^2 \|x\|^2.\]

By Theorem 2, \(K(N) \leq b/a.\)

Lemma 2 on twice-differentiable functions may be used to omit from the hypothesis of the above theorem the condition on the existence of the point \(y\) if the conclusion is altered to read “\(K(N') \leq b/a\) for every two-dimensional subspace \(N'\) of \(N.\)” However, this does not insure that \(K(N) \leq b/a.\)

To get Carlsson’s theorem from Corollary 2.2, define
\[G(t_1, t_2, \ldots, t_n) = -\frac{1}{a_0} \sum_{i=1}^{n} a_i t_i^2\]
and note that \(G\) is twice-differentiable everywhere. Then, using Lemma 2 on twice-differentiable functions, we have that every 2-dimensional subspace of \(N\) is an inner product space, and, hence, that \(N\) is an inner product space.

If \(n=3,\) then the condition that \(G\) be twice-differentiable may be omitted from the hypothesis of Theorem 2, as this author has shown in [3, Theorem 6]. One is tempted to guess that for other values of \(n\) that condition is also unnecessary. The following example disproves that conjecture.

In the plane let \(A\) be the arc \{(a, b):b \geq 0, a^2 + (b + 1/2)^2 = 1\} of the circle with radius 1 and center \((0, -1/2)\). Let \(S\) be the unit sphere \(A \cup (-A)\) and let \(\|\cdot\|\) be the corresponding norm (the Minkowski functional for \(S\)). If \(S'\) is any linear image of \(S\) distinct from \(S,\) then \(S\) and \(S'\) have no more than 3 pairwise linearly independent points in common. Let \((a_i, b_i), i=0, 1, 2, 3, 4,\) be five pairwise linearly independent points. We show that there is a function \(G\) such that, for every two points \(x\) and \(y,\)
\[G(\|a_0 x + b_0 y\|) = a_0 x + b_0 y\|^2.\]

Let \(x\) and \(y\) be points in the plane. By the strict convexity of \(S,\) \(x\) and \(y\) are linearly dependent if and only if there exist numbers \(c\) and \(d\) such that
\|a_i x + b_i y\| = |a_i c + b_i d|, \, i = 1, 2, 3, 4, \text{ and in this case, define}

\[ G(\|a_i x + b_i y\|) = (a_0 c + b_0 d)^2, \]

which is \(\|a_0 x + b_0 y\|^2\). Suppose \(x\) and \(y\) are linearly independent. The four points \((a_i, b_i)/\|a_i x + b_i y\|, \, i = 1, 2, 3, 4, \) belong to \(T(S)\), where \(T\) is the linear mapping such that \(T(x) = (1, 0)\) and \(T(y) = (0, 1)\). There exists no other linear image of \(S\) containing these points. Let \(k\) denote the positive number such that \((a_0, b_0)/k \in T(S)\). Then

\[ 1 = \|T^{-1}((a_0, b_0)/k)\| = \|a_0 x + b_0 y\|/k, \]

so \(k = \|a_0 x + b_0 y\|\). Thus for \(x\) and \(y\) linearly independent, the rule is

\[ G(\|a_0 x + b_0 y\|) = k^2, \]

where \(k\) is the positive number such that \((a_0, b_0)/k\) belongs to the unique linear image of \(S\) which contains \((a_i, b_i)/\|a_i x + b_i y\|, \, i = 1, 2, 3, 4.\) Although such a function \(G\) exists, \(N\) is not an inner product space.

We have used the following two lemmas on twice-differentiable functions. The proof of Lemma 1 is omitted since it is quite similar to the proof of the ordinary composition theorem ("chain rule") in differential calculus.

**Lemma 1.** Suppose that \(F\) is a function from a subset \(A\) of the normed linear space \(N_1\) into the normed linear space \(N_2\), \(x\) is a point of \(A\) where \(F\) is twice-differentiable, \(G\) is a function from a subset of \(N_2\) containing \(F(x)\) into a normed linear space \(N_3\), \(G\) is twice-differentiable at \(F(x)\), and \(x\) is a member and limit point of \(\text{dom } F \circ G\). Let \(S_1\) and \(S_2\) (\(S_1\) linear and \(S_2\) bilinear) be functions satisfying the definition of twice-differentiability for \(F\) at \(x\), and let \(T_1\) and \(T_2\) be functions satisfying the definition for \(G\) at \(F(x)\). Then \(G \circ F\) is twice-differentiable at \(x\) and has derivatives \(U_1 = T_1 \circ S_1\) and \(U_2 = T_1 \circ S_2 + T_2 \circ (S_1, S_1)\).

**Lemma 2.** Every norm defined on the plane is twice-differentiable almost everywhere.

**Proof.** Suppose \(\|\cdot\|\) is a norm defined on the plane. Let \(x\) and \(y\) be two linearly independent points of the unit sphere, and let \(r\) be the positive function such that \(\|r(\theta) \cos(\theta)x + r(\theta) \sin(\theta)y\| = 1\) for all \(\theta\). Then \(r\) is left-differentiable and for every \(\theta\), \(-\arctan(r_\theta'/(r(\theta))) + \theta + \pi/2\) gives the direction of the left-hand tangent to the unit sphere at \(r(\theta) \cos(\theta)x + r(\theta) \sin(\theta)y\), so the function \(-\arctan(r_\theta'/(r(\theta))) + \theta + \pi/2\) is nondecreasing and differentiable almost everywhere. It turns out that \(r\) is twice-differentiable wherever \(r_\theta'\) is differentiable.

Suppose \(r_\theta'\) is differentiable at \(\theta\). Then \(r_\theta'\) is continuous at \(\theta\) and \(r\) is differentiable at \(\theta\). Let \(T_1\) be the linear function and \(T_2\) the bilinear function defined by \(T_1(\varepsilon) = \varepsilon r_\theta'(\theta)\) and \(T_2(\varepsilon, \varepsilon) = \varepsilon^2 (r_\theta'(\theta))\). For every \(\varepsilon\), let
$h(\varepsilon) = r(\theta + \varepsilon) - r(\theta) - T_1(\varepsilon)$. Then $h(0) = h'(0) = 0$, $h$ is left-differentiable, and $h'$ is differentiable at 0. It is a theorem that, under these conditions, \[ \lim_{\varepsilon \to 0} h(\varepsilon)/\varepsilon^2 = \frac{1}{2} h'(0)/2. \] This implies that
\[
\lim_{\varepsilon \to 0} \frac{r(\theta + \varepsilon) - r(\theta) - T_1(\varepsilon) - \frac{1}{2} T_2(\varepsilon, \varepsilon)}{\varepsilon^2} = 0,
\]
so $r$ is twice-differentiable almost everywhere.

The function $f$ from the plane into $\mathbb{R}$ defined by
\[
f(k \cos(\theta)x + k \sin(\theta)y) = \theta
\]
if $k > 0$ and $\theta \in [0, 2\pi)$ is twice-differentiable except on the ray $\{kx: k \geq 0\}$, and, hence, $r \circ f$ and $(r \circ f)^{-1}$ and the function $\|p\| = |p|/(r \circ f(p))$ are twice-differentiable almost everywhere.

**REFERENCES**