THE SOLUTION OF AN INTEGRAL EQUATION

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Abstract. Various methods are developed to solve the integral equation \( f(x) = \int_0^\infty g(t)k(xt) \, dt \), when the Mellin transform \( K(s) \) of the kernel function \( k(x) \) is decomposable. Each method corresponds to the way \( K(s) \) is decomposed: Namely (i) \( K(s) = \frac{1}{L(1-s)}M(1-s) \), (ii) \( K(s) = H(s)M(1-s) \) and (iii) \( K(s) = N(s)H(s) \), where \( L, M, N \) and \( H \) are arbitrary functions of the complex variable \( s \). Numerous special cases and examples are given to illustrate the technique and the advantage of these methods.

1. Introduction. Let us consider the integral equation

\[ g(x) = \int_0^\infty f(t)k(xt) \, dt, \quad x > 0, \]

where the function \( g(x) \) and \( k(x) \) are known and \( f(x) \) is to be found. Many techniques are known to solve this integral equation, depending upon the properties which the Mellin transform of the function \( k(x) \), called the kernel, must satisfy. For example, the most familiar case is the classical one, where \( k(x) \) is the Fourier kernel [2, Chapter III], that is, its Mellin transform \( K(s) \), satisfies the following functional equation

\[ K(s)K(1-s) = 1, \]

where \( s = c + it, -\infty < t < \infty \), for some real \( c \), and \( k(x) \) in some sense is given by

\[ k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s)x^s \, ds. \]

Then, by the Fourier inversion formula, the solution is

\[ f(x) = \int_0^\infty k(xt)g(t) \, dt. \]
If $K(s)$ is a rational function involving gamma function, i.e.

$$K(s) = \prod_{i=1}^{m} \Gamma(\alpha_i s + \beta_i) / \prod_{j=1}^{n} \Gamma(\alpha_j s + \beta_j),$$

then by applying the operators $L$ and $L^{-1}$, to both sides of (1), $L$ being the Laplace operator, the solution can be found [1]. Also, the integral equation (1) is solvable in terms of Mellin transforms and the solution is,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{G(1 - s)}{K(1 - s)} x^s \, ds,$$

where $G(s)$ is the Mellin transform of $g(x)$.

In this paper we shall give a method for solving the integral equation (1), when $K(s)$ is decomposable into arbitrary functions of $s$. The cases which we shall consider here are when $K(s)$ is decomposable into the following forms:

(i) $K(s)=L(1-s)M(1-s)$,
(ii) $K(s)=H(s)M(1-s)$,
(iii) $K(s)=N(s)H(s),$

where $L, M, N$ and $H$ are arbitrary functions of $s; s=c+it, -\infty<t<\infty.$

2. Preliminaries. For simplicity, we shall confine ourselves to $L^2$-space and base the proofs given here upon the Mellin transforms. The following well-known theorems will be used and are found in [2, pp. 94, 95].

A. If $f(x) \in L^2(0, \infty)$, then

$$F(s) = \int_{0}^{\infty} f(t) t^{s-1} \, dt,$$

the integral converging to $F(s)$ in the mean square sense and $F(s) \in L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty).$

If $F(s) \in L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$, then

$$f(x) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s) x^s \, ds,$$

the integral converging to $f(x)$ in the mean square sense and $f(x) \in L^2(0, \infty)$.

This defines $F(s)$ to be the Mellin transform of $f(x)$.

B. The Parseval Theorem. If $f(x)$ and $g(x)$ both belong to $L^2(0, \infty)$, and $F(s), G(s)$ are their respective Mellin transforms, then

$$\int_{0}^{\infty} f(x)g(x) \, dx = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s)G(1-s) \, ds.$$
C. If $f(x)$ and $g(x)$ both belong to $L^2(0, \infty)$, then by the results (A) and (B), the integral equation (1) gives

$$\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} G(s)x^s ds = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(1 - s)K(s)x^s ds.$$  

Therefore

$$\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} [G(s) - F(1 - s)K(s)]x^s ds = 0,$$

and

$$G(s) = F(1 - s)K(s)$$
a.e. on $s = \frac{1}{2} + it$, $-\infty < t < \infty$.

3. Theorems.

**Theorem 1.** If $K(s) = L(1 - s)M(1 - s)$, $s = \frac{1}{2} + it$, $-\infty < t < \infty$, then define $\Phi(s)$ as $\Phi(s) = L(s)G(1 - s)$. Then the solution of the integral equation (1) is

$$f(x) = \int_0^\infty \frac{1}{t} (1/t) m(xt) dt, \quad x > 0,$$

where $K(s)$, $\Phi(s)$ and $M(s)$ are the respective Mellin transforms of the functions $k(x)$, $\varphi(x)$ and $m(x)$, according to the result (A).

**Proof.** Since $\varphi(x) \in L^2(0, \infty)$, by hypothesis, therefore it is an easy matter to see that $(1/x)\varphi(1/x) \in L^2(0, \infty)$ also.

Now $m(x)$ and $(1/x)\varphi(1/x)$ both belong to $L^2(0, \infty)$, therefore the integral $\int_0^\infty (1/t)\varphi(1/t)m(xt) dt$, is absolutely convergent for $x > 0$. By applying the Parseval Theorem (B) for Mellin transforms of $L^2$-functions, we obtain

$$\int_0^\infty \frac{1}{t} \varphi \left( \frac{1}{t} \right) m(xt) dt = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Phi(s)M(s)x^s ds.$$

Now due to the result (C) we have

$$G(s) = F(1 - s)K(s) = F(1 - s)/L(1 - s)M(1 - s).$$

Therefore $F(s) = G(1 - s)L(s)M(s) = \Phi(s)M(s)$. And

$$\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Phi(s)M(s)x^s ds = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(s)x^s ds = f(x),$$
in the mean square sense by the result (A), as required.

**Theorem 2.** If $K(s) = H(s)/M(1 - s)$, $s = \frac{1}{2} + it$, $-\infty < t < \infty$, then define $\Phi(s)$ to be $\Phi(s) = G(1 - s)/H(1 - s)$. Then the solution of the integral equation
(1) is

\[ f(x) = \int_0^\infty \frac{1}{t} \varphi \left( \frac{1}{t} \right) m(xt) \, dt, \quad x > 0, \]

where \( K(s), \Phi(s) \) and \( M(s) \) are as in Theorem 1.

**Proof.** Again \((1/x)\varphi(1/x)\) and \(m(x)\) both belong to \(L^2(0, \infty)\), therefore by the result (B),

\[ \int_0^\infty \frac{1}{t} \varphi \left( \frac{1}{t} \right) m(xt) \, dt = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Phi(s)M(s)x^s \, ds. \]

Also, due to (C),

\[ G(i) = F(1 - s)K(s) = F(1 - s)H(s)/M(1 - s), \]

and therefore

\[ F(s) = G(1 - s)M(s)/H(1 - s) = \Phi(s)M(s). \]

Thus

\[ \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Phi(s)M(s)x^s \, ds = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(s)x^s \, ds = f(x) \]

in the mean square, by the result (A), as required.

In the case when \( K(s) = N(s)H(s) \), if we define \( \phi(s) = G(1-s)H(1-s) \), then it is easy to see that the solution of the integral equation \( \Phi(x) = (1/x)\int_0^\infty f(u)n(u/x) \, du \) is also the solution of (1).

**Note 1.** In Theorem 1, if \( L(s)M(s) = H(s) \), then we have the functional relation \( K(s)H(1-s) = 1 \). The solution of (1) is then given by

\[ f(x) = \int_0^\infty g(t)h(xt) \, dt, \]

which is the usual unsymmetrical Fourier integral formula [2, Chapter VIII]. Numerous examples of the relation between \( h(x) \) and \( k(x) \) can be found in the literature [2, Chapter VIII].

**Note 2.** In Theorem 1, if \( L(s)L(1-s) = 1 \), then \( K(s) = L(s)/M(1-s) \) which is the case considered in Theorem 2.

**Note 3.** In Theorem 2, if \( H(s)H(1-s) = M(s)M(1-s) \), then

\[ K(s)K(1-s) = 1, \]

and then \( k(x) \) is the Fourier kernel for the pair of functions \( f(x) \) and \( g(x) \). Hence the solution of (1) is

\[ f(x) = \int_0^\infty g(t)k(xt) \, dt. \]
The results mentioned above will also hold in the function space $L(0, \infty)$. For this we need to impose appropriate conditions on the functions involved so that the results corresponding to (A) and (B) can be applied. These can be found in [2, pp. 46, 60].

4. Examples. The methods given above to solve the integral equation (1), essentially depend upon the decomposition of $K(s)$. This is done in such a way, so that the following integrals can be evaluated to give, respectively,

(i) $(1/2 \pi i) \int_{1/2-i\infty}^{1/2+i\infty} M(s)\tilde{x}^s \, ds = m(x),$
(ii) $(1/2 \pi i) \int_{1/2}^{1/2} \Phi(s)\tilde{x}^s \, ds = \varphi(x),$ and
(iii) $\int_0^\infty (1/t) \varphi(1/t)m(xt) \, dt = f(x).

The advantage of these methods lies, therefore, in assuming that these above integrals involve elementary functions and are easy to evaluate or can simply be read from the tables ([3], [4]). To confirm this assumption, we give below a few examples. Throughout these examples the Mellin transforms are defined for the appropriate range of Re$(s)$.

1. Laplace's integral equation.

$$g(x) = \int_0^\infty f(t) e^{xt} \, dt.$$ (i) Here $k(x) = \tilde{e}^x$ and $K(s) = \Gamma(s) = \pi/\Gamma(1-s)\sin \pi s$, where $L(s) = \sin \pi s$ and $M(s) = \Gamma(s)$. Thus $m(x) = \tilde{e}^x$.

If $g(x) = \log x/(x-1)$, so that $G(s) = \pi^2 \cosec^2 \pi s$ [4, p. 538], then

$$\Phi(s) = L(s)G(1-s) = \pi \cosec \pi s,$$

whence $\varphi(x) = 1/(1+x)$ [3, p. 308]. Thus by Theorem 1,

$$f(x) = \int_0^\infty \frac{1}{t} \varphi(1/t)m(xt) \, dt = \int_0^\infty \frac{\tilde{e}^{xt}}{1+t} \, dt = \Gamma(0, x)e^x \quad [3, p. 137].$$

2. Stieltjes's integral equation.

$$\Psi(x) = \int_0^\infty \frac{f(u)}{x+u} \, du.$$ By change of variable, we can write this as

$$g(x) = \frac{1}{x} \Psi\left(\frac{1}{x}\right) = \int_0^\infty \frac{f(x)}{1+xt} \, dt.$$ (i) Consider $k(x) = 1/(1+x)$. Then

$$K(s) = \pi \cosec \pi s = \frac{\pi \Gamma(1-s)}{2\Gamma(1-s)\sin \frac{1}{2} \pi s \cos \frac{1}{2} \pi s}.$$
where \( H(s) = \Gamma(1-s)\csc\frac{\pi}{2}(1-s) \) and \( M(s) = -(2/\pi)\Gamma(s)\cos \frac{\pi}{2}s \), thus \( m(x) = -(2/\pi)\cos x \) [3, p. 348].

Now if \( \Psi(x) = K_0(2x^{1/2}) \), then \( g(x) = x^1K_0(2x^{-1/2}) \), and \( G(s) = \Gamma^2(1-s) \), and \( \Phi(s) = G(1-s)/H(1-s) = \Gamma(s)\sin \frac{\pi}{2}s \), whence \( \varphi(x) = \sin x \).

Thus by Theorem 2

\[
f(x) = \int_0^\infty \frac{1}{t} \varphi \left( \frac{1}{t} \right) m(\sqrt{t}x) \, dt = -\frac{2}{\pi} \int_0^\infty \frac{1}{t} \sin \left( \frac{1}{t} \right) \cos x \, dt = -J_0(2x^{1/2})
\]

[3, p. 25].

(ii) Now

\[ K(s) = \pi \csc \pi s = \frac{\pi \csc \pi s \Gamma(1-s)}{\Gamma(1-s)}, \]

where \( H(s) = \pi \Gamma(1-s)\csc \pi s \) and \( M(s) = \Gamma(s) \). Thus \( m(x) = e^x \).

If \( \Psi(s) = \log x/(x-1) \), then \( g(x) = \log x/(x-1) \), and \( G(s) = \pi^2 \csc^2 \pi s \) [4, p. 538].

We define, \( \Phi(s) = G(1-s)/H(1-s) = \Gamma(1-s) \), whence \( \Phi(x) = x^1e^{1/x} \).

Thus by Theorem 2,

\[
f(x) = \int_0^\infty \frac{1}{t} \varphi \left( \frac{1}{t} \right) m(\sqrt{t}x) \, dt = \int_0^\infty e^{(1+x)t} \, dt = \frac{1}{1+x}.
\]

3. Cosine integral equation.

\[
g(x) = \int_0^\infty f(t) \cos xt \, dt.
\]

Here \( k(x) = \cos x \), therefore \( K(s) = \Gamma(s)\cos \frac{\pi}{2}s = \Gamma(s)/\sec \frac{\pi}{2}s \), where \( H(s) = \Gamma(s) \) and \( M(s) = \csc \frac{\pi}{2}s \). Thus \( m(x) = 1/(1+x^2) \) [3, p. 345].

Now, if \( g(x) = 1/(1+x) \), then \( G(s) = \csc \pi s \), and

\[
\Phi(s) = G(1-s)/H(1-s) = \Gamma(s),
\]

whence \( \varphi(x) = e^x \). Thus by Theorem 2,

\[
f(x) = \int_0^\infty \frac{1}{t} \varphi \left( \frac{1}{t} \right) m(\sqrt{t}x) \, dt = \int_0^\infty \frac{1}{u^2 + 1} e^{-u} \, du = ci(x)\cos x - si(x)\sin x \quad [4, p. 312].
\]

4. Let \( k(x) = J_0(u) \), and therefore

\[
K(s) = 2^{s-1} \frac{\Gamma(\frac{1}{2}s + \frac{1}{2}v)}{\Gamma(\frac{1}{2}v - \frac{1}{2}s + 1)}, \quad -R(v) < R(s) < \frac{3}{2},
\]

where

\[
H(s) = 2^{s-1}\Gamma(\frac{1}{2}s + \frac{1}{2}v) \quad \text{and} \quad M(s) = \Gamma(\frac{1}{2}v + \frac{1}{2}s + \frac{1}{2}),
\]
whence \( m(x) = 2x^{v+1}e^{\alpha^2} \). Now if \( g(x) = K_v(x) \), then \( \Phi(s) = G(1-s)/H(1-s) \) gives \( \phi(x) = x^{v-1}e^{-1/2} \). Thus

\[
f(x) = \int_0^\infty \frac{1}{t} \phi \left( \frac{1}{t} \right) m(xt) \, dt = 2x^{v+1} \int_0^\infty t e^{t(1+x^2)} \, dt = \frac{x^{v+1}}{1 + x^2},
\]

\(-1 < R(v) < \frac{3}{2}\).

More generally if \( g(x) = x^\mu K_v(x) \), then \( f(x) = 2^\mu \Gamma(\mu + 1)x^{v+1}/(1+x^2)^{\mu+1} \).

5. Let \( k(x) = x^{1/2}H_v(x) \); then \([3, \text{p. 335}] K(s) = H(s)/M(1-s) \) where

\[
H(s) = \tan \left( \frac{\pi}{2} (s + v + \frac{1}{2}) \right) \text{ and } M(s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}v + \frac{1}{2}s + \frac{1}{2})}{\Gamma(\frac{1}{2}v - \frac{1}{2}s + \frac{1}{2})}
\]

whence \( m(x) = x^{1/2}J_v(x) \).

Now if \( g(x) = x^{v-1/2}/(1+x^2) \), then \( \Phi(s) = G(1-s)/H(1-s) \) gives \( \phi(x) = x^{1/2-v}/(1+x^2) \). Thus by Theorem 1,

\[
f(x) = \int_0^\infty \frac{1}{t} \phi \left( \frac{1}{t} \right) m(xt) \, dt = x^{1/2} \int_0^\infty \frac{t^{v+1}}{1 + t^2} J_v(xt) \, dt = x^{1/2}K_v(x),
\]

\(-1 < R(v) < \frac{3}{2} \) \([4, \text{p. 686}]\).

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