PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 1, September 1973

COMPACT $\mathscr{G}$-SOUSLIN SETS ARE G_δ'S

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ABSTRACT. The result of the title generalizes and places in a new set-theoretical context the well-known theorem of Halmos that every compact Baire set is a $\mathscr{G}_\delta$. Šneĭder's result of 1968 that every perfectly normal compact space with $\mathscr{G}$-Souslin diagonal is metrizable, can now be seen to be true without the perfect normality condition.

A set-theoretic lemma is established and used firstly to show that every countably compact $\mathscr{G}$-Souslin set is a $\mathscr{G}_\delta$ (Theorem 1).

One immediate application of this result is the following. In 1945, V. Šneĭder [8] proved that every compact Hausdorff space with a $\mathscr{G}_\delta$ diagonal is metrizable, and in 1968 (see [9] and [10]) obtained the same conclusion for perfectly normal spaces with $\mathscr{G}$-Souslin diagonals. His 1968 result can now be seen to follow without the perfect normality condition from his work in 1945.

A second area of application of Theorem 1 stems from a result of Halmos [4, Theorem D, p. 221] viz. "Every compact Baire set is a $\mathscr{G}_\delta$." Several generalizations have been established (see, for example, [1] and [7]), mostly in the direction of abstractions of Halmos' proof. For example, every Baire set is "distinguishable" and every compact distinguishable set is a $\mathscr{G}_\delta$ (see [3]). Since every Baire set is $\mathscr{G}$-Souslin and $\mathscr{G}$-Souslin sets are not necessarily distinguishable, Theorem 1 is a new (and purely set theoretical) generalization of Halmos' result. That of C. A. Rogers in [6] follows by a similar application of the lemma.

The Lemma is easily applied to show that every compact subset of a T_1 space with infinite weight $\mathfrak{n}$ is the intersection of $\mathfrak{n}$ open sets. A similar result holds for $\mathfrak{k}$-Lindelöf sets.

The author is indebted to the referee for comments on the original version of this paper.

Notation. The *diagonal* of a set X is defined as $\{(x, x): x \in X\}$. Let N be the set of positive integers, and let

$$S = \{(n_1, n_2, \cdots, n_k): k, n_1, n_2, \cdots, n_k \in N\}.$$

Received by the editors May 10, 1972.

AMS (MOS) subject classifications (1970). Primary 02K30, 04A15, 28A05, 54H05, 04-00, 54D10, 54A25.

Key words and phrases. Souslin, compact, $\mathscr{G}_\delta$, second countable.

For $\mathscr{H} \subseteq \exp X$, an *$\mathscr{H}$-Souslin* set is a set of the form

$$\bigcup \{\bigcap \{H(s): s < \sigma\}: \sigma \in N^N\} \quad \text{where} \quad H: S \to \mathscr{H},$$

and "$s<(t_1, t_2, \cdots)$" means that $s=(t_1, t_2, \cdots, t_n)$ for some n. A subspace V of a topological space X is *$\mathfrak{k}$-Lindelöf* for a cardinal number $\mathfrak{k}$ if every open cover of V contains a subcover with cardinality less than or equal to $\mathfrak{k}$. (Thus, $\aleph_0$-Lindelöf=Lindelöf.) A $\mathscr{H}_\mathfrak{n}$ is an intersection of $\mathfrak{n}$ or fewer elements of $\mathscr{H}$. We denote by $\mathscr{F}(X)$ [$\mathscr{G}(X)$]—or simply $\mathscr{F}$ [$\mathscr{G}$] when unambiguous—the family of closed (open) sets of X. (Thus "$\mathscr{G}_{\aleph_0}$"="$\mathscr{G}_\delta$".) By V' we mean the complement of V.

LEMMA. *Let X be a set, V a subset, $\mathscr{U}$ a cover of V and $\boldsymbol{W}$ the collection of subcovers of $\mathscr{U}$. Suppose that $\boldsymbol{B} \subseteq \boldsymbol{W}$ and $\boldsymbol{B}$ contains a subcover of every element of $\boldsymbol{W}$. Set $L=\bigcap \{\bigcup \mathscr{M}: \mathscr{M} \in \boldsymbol{B}\}$.*

Then $P \supseteq L$ if, and only if, $P \supseteq V$ and for every $(x, x_1) \in V \times P'$ there exists $U \in \mathscr{U}$ such that $(x, x_1) \in U \times U'$.

PROOF. *Sufficiency.* Suppose that P satisfies the given condition. Let $x_1 \in P'$, and for each x in V, let $U_x \in \mathscr{U}$ be such that $(x, x_1) \in U_x \times U_x'$. Now $\{U_x: x \in V\} \in \boldsymbol{W}$ and therefore has a subcover $\mathscr{J} \in \boldsymbol{B}$. Thus $x_1 \notin \bigcup \{U_x: x \in V\} \supseteq \bigcup \mathscr{J} \supseteq L$, and so P contains L.

Necessity. Note that $V \subseteq L \subseteq P$. Let $(x, x_1) \in V \times P'$. Since $x_1 \notin L$, we have $x_1 \notin \bigcup \mathscr{M}$ for some $\mathscr{M} \in \boldsymbol{B} \subseteq \boldsymbol{W}$, and the result follows.

THEOREM 1. *Let G be a $\mathscr{G}$-Souslin set of a topological space X, and K a countably compact subset of G. There exists a $\mathscr{G}_\delta$ set L with $K \subseteq L \subseteq G$.*

PROOF. Let $G=\bigcup \{\bigcap \{U(s): s<\sigma\}: \sigma \in N^N\}$, with $U(s)$ open for each s in S. If $(x, x_1) \in K \times G'$, there exists $s \in S$ for which $(x, x_1) \in U(s) \times U(s)'$. We can thus apply the Lemma with $V=K$, $\mathscr{U}=\{U(s): s \in S\}$, $\boldsymbol{B}= \{\mathscr{M} \in \boldsymbol{W}: \mathscr{M}$ is finite$\}$ and $P=G$, concluding that

$$K \subseteq L = \bigcap \{\bigcup \mathscr{M}: \mathscr{M} \subseteq \mathscr{U} \text{ is a finite cover of } K\} \subseteq G.$$

The set L is a $\mathscr{G}_\delta$.

COROLLARIES. (1.1) *Every countably compact $\mathscr{G}$-Souslin set is a $\mathscr{G}_\delta$.*

(1.2) *A compact Hausdorff space X is metrizable if, and only if, its diagonal is $\mathscr{G}$-Souslin.*

(1.3) *The families of $\mathscr{F}$-Souslin sets and $\mathscr{G}$-Souslin sets of a countably compact Hausdorff space coincide if, and only if, every closed set is a $\mathscr{G}_\delta$.*

PROOFS. Part 1 follows directly from Theorem 1.

Part 2 is proved in the introduction.

Part 3 is proved as follows: If the two families coincide, then every (countably compact) closed set is $\mathscr{G}$-Souslin and therefore a $\mathscr{G}_\delta$ by

Theorem 1. If every closed set is a $\mathscr{G}_\delta$, every $\mathscr{F}$-Souslin set is $\mathscr{G}$-Souslin (see, for example, [**5**, p. 106]). Similarly, every $\mathscr{G}$-Souslin set is $\mathscr{F}$-Souslin.

THEOREM 2. *Let X be a T_1 space with weight* $\mathfrak{m}$.

(a) *Every compact subset of X is a $\mathscr{G}_{\mathfrak{m}}$ whenever* $\mathfrak{m}$ *is infinite.*

(b) *Every $\mathfrak{k}$-Lindelöf subset of X is a $\mathscr{G}_{\mathfrak{m}}$ whenever* $\mathfrak{m}^{\mathfrak{k}}=\mathfrak{m}$.

PROOF. Let $V\subseteq X$ and let $\mathscr{U}$ be a basis with cardinality $\mathfrak{m}$. In both cases we apply the "sufficiency" part of the Lemma with $P=V$, concluding that

$$(\#) \qquad V = L = \bigcap \{\bigcup \mathscr{M} : \mathscr{M} \in \boldsymbol{B}\}$$

for every family $\boldsymbol{B}$ containing subcovers of all elements of $\boldsymbol{W}$.

The equality (#) yields the conclusions of Theorem 2 by taking $\boldsymbol{B}$ to be, respectively,

(a) the family of finite subcovers of $\mathscr{U}$ and (b) the family of subcovers of $\mathscr{U}$ with cardinality less than or equal to $\mathfrak{k}$.

Notes. 1. Topological spaces whose weight $\mathfrak{m}$ satisfies $\mathfrak{m}^{\mathfrak{k}}=\mathfrak{m}$ are discussed, for example, in [**2**].

2. Theorem 2 has the trivial converse that if X is a topological space satisfying (a) or (b) for some $\mathfrak{m}$, then X is T_1.

The following corollary follows directly from Theorem 2(a).

COROLLARY 2. *Every compact subspace of a second countable T_1 space is a $\mathscr{G}_\delta$.*

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