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EXTENDING A JORDAN RING HOMOMORPHISM

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ABSTRACT. In this paper a homomorphism from an ideal $\mathfrak{B}$ of a quadratic Jordan algebra $\mathfrak{J}$ without 2-torsion over a ring Φ onto a unital quadratic Jordan algebra $\mathfrak{J}'$ without 2-torsion is extended to a homomorphism from $\mathfrak{J}$ to $\mathfrak{J}'$. We then show if D is any class of quadratic Jordan algebras without 2-torsion, then the upper radical property determined by D is hereditary.

1. **Preliminaries.** We adopt the notation and terminology of an earlier paper [3] concerning quadratic Jordan algebras and hereditary radical properties. For a discussion of *upper radical properties* the reader is referred to [1] and [2]. Basically, given a class of rings D with the property that any nonzero ideal of a ring of D can be mapped homomorphically onto a ring of D, then D can be extended to a class of all D semisimple rings: a ring is D *radical* if it cannot be mapped homomorphically onto a ring of D; a ring without D radical ideals is D *semisimple*.

2. **Two theorems.**

THEOREM 1. *Given a quadratic Jordan algebra $\mathfrak{J}$ with ideal $\mathfrak{B}$ and a unital quadratic Jordan algebra $\mathfrak{J}'$ without 2-torsion, a homomorphism φ from $\mathfrak{B}$ onto $\mathfrak{J}'$ can be extended to a homomorphism $\tilde{\varphi}:\mathfrak{J}\to\mathfrak{J}'$.*

PROOF. Since φ is an onto homomorphism, some element of $\mathfrak{B}$, say b, maps into 1. Define $\tilde{\varphi}:\mathfrak{J}\to\mathfrak{J}'$ by $\tilde{\varphi}(a)=\varphi(U_b a)$. $\tilde{\varphi}$ is clearly linear. For $b'\in\mathfrak{B}$,

$$\tilde{\varphi}(b') = \varphi(U_b b') = U_{\varphi(b)}\varphi(b') = U_1\varphi(b') = \varphi(b').$$

It is therefore established that $\tilde{\varphi}$ extends φ. It now remains to show that $\tilde{\varphi}$ is indeed a homomorphism. Since $2U_x=V_x^2-V_{x^2}$, under the assumption that $\mathfrak{J}'$ has no 2-torsion, it is sufficient to show that $\tilde{\varphi}$ preserves the V operator, i.e. to show $\tilde{\varphi}(a\circ a')=\tilde{\varphi}(a)\circ\tilde{\varphi}(a')$ for a, $a'\in\mathfrak{J}$. For then $\tilde{\varphi}(2U_x y)=2U_{\tilde{\varphi}(x)}\tilde{\varphi}(y)$ and $2\{\tilde{\varphi}(U_x y)-U_{\tilde{\varphi}(x)}\tilde{\varphi}(y)\}=0$ implies $\tilde{\varphi}(U_x y)-U_{\tilde{\varphi}(x)}\tilde{\varphi}(y)=0$. First a simplification: it will be convenient to express $U_b U_b V_a a'$ as

$$U_b[(b\circ a)\circ(b\circ a')] - (U_b a)\circ(U_b a') - b\circ\{b\, a\, U_b a'\},$$

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an identity which is easily verified in the case of special Jordan algebras and is therefore true for all Jordan algebras as a consequence of Macdonald's theorem [4]. Now, since $\tilde{\varphi}(b')=\varphi(b')=\varphi(U_b b')$ for $b' \in \mathfrak{B}$,

$$\begin{aligned}\tilde{\varphi}(a \circ a') &= \varphi(U_b(a \circ a')) = \varphi(U_b U_b(a \circ a')) \\ &= \varphi(U_b[(b \circ a) \circ (b \circ a')] - (U_b a) \circ (U_b a') - b \circ \{b\, a\, U_b a'\}).\end{aligned}$$

Consider these terms one at a time.

$$\begin{aligned}\varphi(U_b[(b \circ a) \circ (b \circ a')]) &= \varphi(U_b[(V_b a) \circ (V_b a')]) = \varphi[(V_b a) \circ (V_b a')] \\ &= \varphi(V_b a) \circ \varphi(V_b a') = \varphi(U_b V_b a) \circ \varphi(U_b V_b a') = \varphi(V_b U_b a) \circ \varphi(V_b U_b a') \\ &= V_1 \varphi(U_b a) \circ V_1 \varphi(U_b a') = 2\tilde{\varphi}(a) \circ 2\tilde{\varphi}(a') = 4(\tilde{\varphi}(a) \circ \tilde{\varphi}(a')).\end{aligned}$$

Next,

$$\varphi[(U_b a) \circ (U_b a')] = \varphi(U_b a) \circ \varphi(U_b a') = \tilde{\varphi}(a) \circ \tilde{\varphi}(a').$$

Finally,

$$\begin{aligned}\varphi(b \circ \{b\, a\, U_b a'\}) &= \varphi(V_b V_{b,a} U_b a') = \varphi(V_b U_{b,U(b)a'} a) \\ &= \varphi[U_{b,U(b)a}(b \circ a') + U_{b,U(b)a'}(b \circ a) - 2(U_b a) \circ (U_b a')] \\ &\qquad\qquad \text{(by Macdonald's theorem)} \\ &= \varphi[U_{b,U(b)a}(b \circ a')] + \varphi[U_{b,U(b)a'}(b \circ a)] - 2[\varphi(U_b a) \circ \varphi(U_b a')] \\ &= \varphi[U_{b,U(b)U(b)a}(b \circ a')] + \varphi[U_{b,U(b)U(b)a'}(b \circ a)] - 2[\tilde{\varphi}(a) \circ \tilde{\varphi}(a')] \\ &= \varphi\{U_b[(U_b a) \circ (U_b a')] + (U_b a') \circ (U_b U_b a)\} \\ &\quad + \varphi\{U_b[(U_b a) \circ (U_b a')] + (U_b a) \circ (U_b U_b a')\} - 2(\tilde{\varphi}(a) \circ \tilde{\varphi}(a')) \\ &\qquad\qquad \text{(again using Macdonald's Theorem)} \\ &= 2(\tilde{\varphi}(a) \circ \tilde{\varphi}(a')).\end{aligned}$$

Combining these results, $\tilde{\varphi}(a \circ a')=\tilde{\varphi}(a) \circ \tilde{\varphi}(a')$. $\tilde{\varphi}$ is therefore a homomorphism and the proof of the theorem is complete.

THEOREM 2. *If D is any class of quadratic Jordan rings with unit element over a ring Φ and no 2-torsion, satisfying the condition that any nonzero ideal of a ring of D can be mapped homomorphically onto a ring of D, then the upper radical property determined by D is hereditary.*

PROOF. Let $\mathfrak{J}$ be a D radical ring and let $\mathfrak{B}$ be a nonzero ideal of $\mathfrak{J}$. Assume $\mathfrak{B}$ is not a D radical ring. Then $\mathfrak{B}$ can be mapped homomorphically onto some ring $\mathfrak{S}$ in the class D. By the above theorem $\mathfrak{J}$ is also homomorphic to $\mathfrak{S}$ which contradicts $\mathfrak{J}$ being D radical.

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