DECOMPOSABLE AND SPECTRAL OPERATORS ON A HILBERT SPACE

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ABSTRACT. A necessary and sufficient condition for a decomposable operator, on a Hilbert space, to be equal to a normal plus a commuting quasi-nilpotent operator is found.

The notion of a decomposable operator on a Banach space was introduced by Foias [6]. It can be viewed as a nontrivial generalization of the notion of a spectral operator, which is due to Dunford ([2], [3], [4]). The question arises: Under what conditions is a decomposable operator a spectral operator? We do not know the complete answer to this question, but in this note we find a condition which is necessary and sufficient for a decomposable operator on a Hilbert space to be equal to a normal operator plus a commuting quasi-nilpotent operator.

Let $T$ be a bounded linear operator on a Hilbert space $H$. A subspace $M$ of $H$ is called a spectral maximal space of $T$ if (i) $M$ is invariant under $T$ and (ii) if $N$ is any subspace of $H$ invariant under $T$ such that $\sigma(T|N) \subseteq \sigma(T|M)$ then $N \subseteq M$, where $T|N$ denotes the restriction of $T$ to the subspace $N$ and $\sigma(T)$ denotes the spectrum of $T$. $T$ is called a decomposable operator if, for every finite open covering $G_i$, $i=1, 2, \ldots, n$, of $\sigma(T)$, there exists a family $M_i$, $i=1, 2, \ldots, n$, of spectral maximal spaces of $T$ such that (i) $H = M_1 + M_2 + \cdots + M_n$, and (ii) $\sigma(T|M_i) \subseteq G_i$ for $i=1, 2, \ldots, n$. These definitions are due to Foias [6]. Let $x \in H$; then $(T-zI)^{-1}x = R(z, x)$ is an analytic vector-valued function for $z \in \text{complement of } \sigma(T) = \rho(T)$. A vector-valued function $f(z)$ is an analytic extension of $R(z, x)$ if it is defined and is analytic on an open set $D(f)$ containing $\rho(T)$ and if $(T-zI)f(z) = x$ for all $z \in D(f)$. $R(z, x)$ possesses the single-valued extension property if any two extensions of $R(z, x)$ coincide on the common domain. If $R(z, x)$ has the single-valued extension property then we define a maximal single-valued extension of $R(z, x)$ by taking the union of all extensions of $R(z, x)$ and we designate this by $R_e(z, x)$. The operator $T$ is said to have the single-valued extension property if $R(z, x)$ has the single-valued extension...
property for all \( x \in H \). We now define

\[
\rho(T, x) = \{ z : \text{Re}(z, x) \text{ is analytic at } z \},
\]

\[
\sigma(T, x)' = [\rho(T, x)]', \quad \text{the complement of } \rho(T, x).
\]

The definitions of \( R_\delta(z, x) \), \( \rho(T, x) \), \( \sigma(T, x) \) are due to Dunford ([2], [3], [4]); the reader will also find a discussion of the properties of \( \sigma(T, x) \) there.

The following result is due to Foias [6], and for more results about decomposable operators the reader is referred to Colojoara and Foias [1].

**Theorem F (Foias).** If \( T \) is a decomposable operator on \( H \), then:

(i) \( T \) has the single-valued extension property.

(ii) For any \( \delta \subseteq \sigma(T) \), \( \delta \) closed, \( M_\delta = \{ x \in H : \sigma(T, x) = \delta \} \) is a spectral maximal space of \( T \), and conversely if \( M \) is any spectral maximal space of \( T \), then \( M = M_{\delta} \) where \( \delta = \sigma(T|M) \).

A decomposable operator \( T \) on a Hilbert space \( H \) is said to possess property (I) if, for every closed set \( \delta \), \( \sigma(T, P_\delta x) \subseteq \sigma(T, x) \) for all \( x \in H \), where \( P_\delta \) denotes the projection of \( H \) onto \( M_\delta \).

**Proposition.** Let \( T \) be a decomposable operator with property (I), then \( \sigma(T, x) \cap \sigma(T, y) = \emptyset \) implies that \( (x, y) = 0 \).

**Proof.** Let \( \sigma(T, x) = \delta_1 \) and \( \sigma(T, y) = \delta_2 \). Since \( T \) has the property (I), \( \sigma(T, P_\delta x) \subseteq \sigma(T, x) = \delta_1 \). But \( P_\delta x \in M_\delta \) and hence \( \sigma(T, P_\delta x) \subseteq \delta_1 \cap \delta_2 = \emptyset \). Thus \( P_\delta x = 0 \) and hence \( (x, y) = 0 \).

The property \( \sigma(T, x) \cap \sigma(T, y) = \emptyset \) implies \( (x, y) = 0 \) is due to Stampfli [9] and is the orthogonality version of Dunford's boundedness condition (B) [4]. If \( T = N + Q \) where \( N \) is normal and \( Q \) is quasi-nilpotent such that \( QN = NQ \), then \( T \) has property (I) (since in this case \( P_\delta \) will commute with \( T \)).

In order to prove our main result we need the following Lemma.

**Lemma.** Let \( T \) be a decomposable operator with property (I). Then for every closed set \( \delta \), \( M_\delta^\perp = \overline{M_\delta} \), where \( \delta' \) is the complement of \( \delta \).

**Proof.** First we shall show that, for any open set \( G \) containing \( \delta \), \( M_\delta^\perp \subseteq M_{\delta'} \). Since \( G \) and \( \delta' \) cover \( \sigma(T) \) and \( T \) is decomposable, there exist spectral maximal spaces \( Z_1 \) and \( Z_2 \) of \( T \) such that \( H = Z_1 + Z_2 \) and \( \sigma(T|Z_1) \subseteq G \cap \sigma(T) \) and \( \sigma(T|Z_2) \subseteq \delta' \cap \sigma(T) \). Let \( x \in M_\delta \) and let \( x = z_1 + z_2 \) where \( z_i \in Z_i \). Since \( Z_1 \subseteq M_\delta^\perp \), \( 0 = P_\delta x = z_1 + P_\delta z_2 \). Thus \( \sigma(T, z_1) = \sigma(T, P_\delta z_2) \subseteq \overline{G} \cap \sigma(T, z_2) \) (because \( T \) has the property (I)). Hence \( \sigma(T, x) \subseteq \sigma(T, z_1) \cup \sigma(T, z_2) \subseteq \delta' \). Hence \( M_\delta^\perp \subseteq M_{\delta'} \) for any open set \( G \supseteq \delta \). Now by Foias [7], \( M_\delta = \bigcap M_\delta \), where the intersection is taken over all open sets \( G \) containing \( \delta \). Hence \( M_\delta^\perp \subseteq M_\delta \) and by the proposition \( \overline{M_\delta} \subseteq M_\delta^\perp \).

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Theorem. Let $T$ be a decomposable operator with property (I). Then $T = N + Q$ where $N$ is normal, $Q$ is quasi-nilpotent and $NQ = QN$.

Proof. For this proof we shall use the notation and the terminology of Dunford and Schwartz [5, pp. 2136–2149]. From the Theorem F and the Proposition it follows that $T$ satisfies Dunford’s conditions (A), (B) and (C). In order to show that $T$ satisfies condition (D), from the Lemma it follows that for any closed set $\delta$, $H = M_{\delta} \oplus \overline{M}_{\delta}$ and hence $\delta \in S_1(T)$. Since $\delta'$ is open, there is an increasing sequence of closed sets $\beta_n$ such that $\delta' = \bigcup \beta_n$. For any $x \in M_{\delta}$, since $\sigma(T, x)$ is compact, $\sigma(T, x) \subset \beta_n$ for large values of $n$. Hence each closed set $\delta$ is in $S(T)$. Now by Theorem XVI.4.5 [5], $T$ is a spectral operator and, by Stampfli [9, Lemma 7], $T$ has a resolution of identity consisting of selfadjoint projections, i.e., $T = N + Q$ where $N$ is normal, $Q$ is quasi-nilpotent and $NQ = QN$.

Remarks. The author, in [10], proved by a different method, that a hyponormal decomposable operator with property (I) is normal. This result now easily follows from the Theorem. Putnam [8, p. 476] has shown the existence of hyponormal nonnormal operators with totally disconnected spectrum. It is a fact that operators with totally disconnected spectrum are decomposable. Thus a decomposable hyponormal operator is not necessarily normal.

References


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