AN EXACT SEQUENCE CALCULATION FOR THE SECOND HOMOTOPY OF A KNOT. II

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ABSTRACT. This paper deals with the computation of the second homotopy of a knot as a module over its fundamental group.

The present note is a continuation of [18] whose notation and bibliography we use throughout.

4. Semidirect products. From (1) we conclude that $\Pi$ is the semidirect product

$$\Pi = \tilde{\Pi} \times_\varphi Z$$

where $\varphi: Z \to \text{Aut} \tilde{\Pi}$ can be described as follows: By (3),

$$\Pi = \cdots \ast_H G^{(-1)} \ast_H G^{(0)} \ast_H G^{(1)} \ast_H \cdots \quad (G^{(j)} \cong G, j \in Z)$$

let $\varphi(r): \Pi \to \tilde{\Pi}$ be the automorphism defined by $g_j \mapsto g_{j+r}$ for $g_j \in G^{(j)}$.

Thus $\Pi$ can be thought of as the group of the $(g, t') (g \in \Pi, j \in Z)$ with multiplication $(g, t')(h, t'') = (\varphi(s)(g) \cdot h, t'+s)$.

Let $\Gamma$ (resp. $\tilde{\Gamma}$) be the integral group ring of $\Pi$ (resp. $\tilde{\Pi}$); then

$$(12') \quad \Gamma = \tilde{\Gamma} \times_\varphi \Lambda$$

where $\times_\varphi$ indicates twisted tensor product [20]. $\Gamma$ can be described as $\tilde{\Gamma} \otimes \Lambda$ with relations $g_j \otimes t' = g_{j+r} \otimes 1$ for $g_j \in G^{(j)}$ and $r \in Z$.

With the notation of §0, the inclusion $Y(0) \subset \mathcal{X}$ induces homomorphisms $G \to \tilde{\Pi}$ and $\pi_2(Y) \to \pi_2(X)$. The first one defines a structure of $G$-module on $\tilde{\Gamma}$ and, by (12'), on $\Gamma$. Recall $\pi_2(Y)$ is a $G$-module (cf. [4, p. 125]) which allows us to construct $\pi_2(Y) \otimes_G \Gamma$. The induced map of second homotopy induces in turn a homomorphism $e: \pi_2(Y) \otimes_G \Gamma \to \pi_2(X)$.

5. The elements of $\pi_2(X)$. Suppose $k: S^n \to S^{n+2}$ is a differentiable knot which admits a minimal Seifert manifold $V$ in the sense of Lemma 1.

**Lemma 5.** The map $e: \pi_2(Y) \otimes_G \Gamma \to \pi_2(X)$, defined by inclusion $Y(0) \subset \mathcal{X}$, is an epimorphism of $\Gamma$-modules.
Proof. We can represent $\sigma \in \pi_2(\tilde{X})$ by a map $f : S^2 \to \tilde{X}$ which is an embedding if $n \geq 3$ or whose self-intersection points do not belong to $\tilde{V}_i = V_0(j+1)= V_1(j)$ in the notation of [18, §0]. The $\tilde{V}_i$ are the liftings to $\tilde{X}$ of the Seifert manifold $V$ [7]. Let $\bar{V} = \bigcup \tilde{V}_i$ and $M = f^{-1}(\bar{V})$ a union of circles in $S^2$ and let $C$ be an innermost [6] component of $M$; $f(C)$ bounds a disk $f(D)$ in, say, $Y(j_0)$ and lies in $V_i(j_0)$ for $t=0$ or 1. Then $C$ determines an element in $\ker \nu_i$. Since $V$ is a minimal manifold, $C$ bounds a disk $D'$ in $V_i(j_0)$. We can “pinch” $\sigma$ along $D'$ to obtain a new representative of $\sigma$ of the form $\sigma' + \gamma \sigma''$ where $\sigma'' \in \pi_2(Y)$ and $\gamma \in \tilde{\Gamma}$. The element $\sigma'$ can be represented by a map $f' : S^2 \to \tilde{X}$ and $f'^{-1}(\bar{V})$ has one less innermost component. By our definitions and by induction, the proof is now complete.

6. Relations in $\pi_2(X)$. We now want to find $K = \ker(\varepsilon)$. Let $x = \sum_i \sigma_i \otimes \gamma_i \in \pi_2(Y) \otimes_G \Gamma$ be in $K$; we can find a map $f : D^3 \to \tilde{X}$ such that the pinching process of §5 applied on $f|D^3$ yields $x$. For $n \geq 5$, $f$ can be taken to be an embedding; for $n=2, 3, 4$ the self-intersections can be made transversal to $\bar{V}$. Let $N = f^{-1}(\bar{V})$ a disjoint union of orientable surfaces $F_j$ with (circular) boundary, properly embedded in $D^3$. Suppose $\partial F_j \subset \partial D^3$ is innermost (i.e., it bounds a 2-disk $\Delta$ in $\partial D^3$); let $\alpha \in \pi_1(F_j)$ be a generator of the fundamental group. Then, since $\nu(x) = 0$ for some $t$, the loop $\alpha \subset \tilde{V}_i$ bounds a disk $\Delta_1$ in $Y(j_0) \cap f(D^3)$ and, since $V$ is minimal, a disk $\Delta_0$ in $\tilde{V}_i$. A homotopy that pulls $\Delta_1$ down to $\tilde{V}_i$ can be used to drag $f(D^3)$ along. This alters $f$ in such a way that a surgery is done to $F_j$ along $\alpha$. After finitely many steps $F_j$ becomes a disk $\Delta_2$. The sphere $f(\Delta) \cup f(\Delta_2) \subset Y(j_0)$ (identified along $f(\partial F_j)$) represents an element $x'$ of $\pi_2(Y(j_0))$ which deforms to $\Delta_0 \cup \Delta_2 = \sigma$, an element of $\pi_2(V_i)$.

Thus $x$ can be deformed to an element of the form $x' + x''$, where $x'$ has been defined above and where $x''$ can be described by a map $f'' : D^3 \to X$ with image disjoint from $Y(j_0)$. We can pull $x'$ down to $Y(j_0-1)$ by noticing that $x' - \nu_1(\sigma) = 0$ in $\pi_2(Y(j_0))$ and that $\nu_0(\sigma) = t = \nu_1(\sigma) \otimes 1$; so,

$$x = x'' + (\nu_0(\sigma) \otimes t - \nu_1(\sigma) \otimes 1).$$

Define $\pi_2(V) \otimes_H \Gamma$ by noticing that $\pi_2(V)$ is an $H$-module and that $\nu_0 : H \to G \to \Gamma$ defines an $H$-module structure on $\Gamma$ by means of the formula $h \gamma = \nu_0(h) \gamma$. If we use $\nu_1$ instead, we get $t \nu_0(\sigma) = \nu_1(\sigma) \otimes \gamma$. Define now $d : \pi_2(V) \otimes_H \Gamma \to \pi_2(Y) \otimes_G \Gamma$ by $d(\sigma \otimes \gamma) = \nu_0(\sigma) \otimes t \gamma - \nu_1(\sigma) \otimes \gamma$, a homomorphism of $\Gamma$-modules.

Theorem 6. The sequence

$$\pi_2(V) \otimes_H \Gamma \xrightarrow{d} \pi_2(Y) \otimes_G \Gamma \xrightarrow{e} \pi_2(X) \longrightarrow 0$$

is an exact sequence of $\Gamma$-modules.
In fact, by the discussion above, if \( x \in K \), \( x \) is a \( \Gamma \)-linear combination of elements of the form \( d(\sigma \otimes 1) \).

7. Examples. I. Let \( k : S^1 \to S^3 \) be a knot and \( F \) a minimal Seifert surface (cf. [9, Chapter IV]). Let \( K : S^2 \to S^4 \) be the 2-knot obtained by spinning [8]. The 3-manifold \( V \) obtained from \( F \) by spinning is a minimal Seifert manifold for \( K \). Similarly \( Y \) obtained from \( S^4 \) by cutting along \( V \) is obtained from \( S^3 - F = Y_0 \) by spinning. Since \( Y_0 \) is aspherical [19], it is clear that \( \pi_2(Y) \) is a free \( G \)-module (cf. [16, p. 216]). Similarly \( \pi_2(V) \) is a free \( H \)-module. The rank of both is twice the genus \( g \) of \( k \). The \( \Gamma \)-modules \( \pi_2(V) \otimes_H \Gamma \) and \( \pi_2(Y) \otimes_G \Gamma \) are then free \( \Gamma \)-modules of rank \( 2g \). Let \( \alpha_1, \cdots, \alpha_g \) be generators of \( \pi_2(K) \) (resp. \( b_1, \cdots, b_g \)) be generators of \( \pi_2(V) \) (resp. of \( G \)); the \( \alpha_i \) are in a one-to-one correspondence with the generators of \( \pi_1(F) \). By (12), the fundamental group of the knot is presented by

\[
\langle t, \alpha_1, \cdots, \alpha_g \mid R_i = 1, i = 1, \cdots, 2g \rangle
\]

where \( R_i = t \alpha_i^{-1} (v_i(a_i))^{-1} \). The matrix of the map \( d \) defined in §6 is then the Jacobian [17, p. 125] of (13), i.e., the matrix \( \| \partial R_i / \partial b_j \| \). This follows at once from the discussion on the Jacobian in [9, p. 42]. This is a new proof of Theorem 2 of [8].

II. As in [12, II], let \( (B^5, B^3) \) be the unknotted ball pair, \( \partial B^3 \) bounds a disk \( \Delta \) in \( \partial B^5 \). Attach a 1-handle \( h^1 \) to \( \partial B^5 - \Delta \). Let \( L = (B^5 - B^3) \cup h^1 \); \( \partial L \) has the homotopy type of \( S^1 \vee S^1 \vee S^3 \). Let \( \alpha \) be the generator of \( \pi_1(\partial L) \) which goes around the handle and \( \beta \) the meridian of \( \partial B^3 \cup \partial B^5 \). The loop \( \beta \) pierces \( \Delta \) at one point. Attach a 2-handle \( h^2 \) along \( \alpha^2 \beta \alpha^{-1} \beta^{-1} \). We know \( \partial(L \cup h^2) \) is the complement of a (slice) knot \( k : S^2 \to S^4 \) and a minimal Seifert manifold for \( k \) can be obtained from \( \Delta \) by attaching a tube so that \( \Delta \cup (\text{tube}) \) misses \( \alpha^2 \beta \alpha^{-1} \beta^{-1} \). The Seifert manifold is thus obtained from \( S^1 \times S^3 \) by punching a hole to it.

Thus \( H = \mathbb{Z} \sigma \) and \( G = \mathbb{Z} \alpha ; \nu_1 \) is given by multiplication by \( t + 1 \) \((t = 0, 1) \). On the other hand \( \pi_2(V) = \mathbb{Z}[H] \) and \( \pi_2(Y) = \mathbb{Z}[G] \). We can choose \( \nu_0(1) = 1 \); then, by the argument in [12, p. 234], \( 2
\nu_0(1) - \alpha^{-1} \nu_1(1) = 0 \). By Theorem 6, the sequence

\[
\Gamma \xrightarrow{d} \Gamma \xrightarrow{\pi_2(X)} 0
\]

where \( d(1) = \beta - 2 \alpha \), is an exact sequence of \( \Gamma \)-modules. Then \( \pi_2(X) = \mathbb{Z}_{(2)} \), where \( \alpha \) acts trivially and \( \beta^{-1} \) acts by multiplication by two (cf. [2, p. 130]).

III. In the above example, both \( \pi_2(V)_H \) and \( \pi_2(Y)_G \) are infinite cyclic and \( \nu_i \) is given by multiplication by \( t + 1 \); similarly, if \( \Pi \) is a knot group with deficiency one, the construction of [12] allows us to find a knot with minimal Seifert manifold obtained by adding 1-handles to a 3-disk. Then
both $\pi_2(V)_H$ and $\pi_2(Y)_G$ are free abelian and the map $\nu_0 - \nu_1$ is unimodular. By Lemma 3, $H_3(\Pi) = 0$.

IV. Observe that the sequence of Theorem 6 yields, by cancellation of the actions of the groups $H$ and $G$, the sequence (2) of Theorem 0. In example III, since $\alpha$ acts trivially, $\pi_2(X) = \pi_2(X)_H$.

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REFERENCES


