AN EXACT SEQUENCE CALCULATION FOR THE SECOND HOMOTOPY OF A KNOT. II

M. A. GUTIÉRREZ

Abstract. This paper deals with the computation of the second homotopy of a knot as a module over its fundamental group.

The present note is a continuation of [18] whose notation and bibliography we use throughout.

4. Semidirect products. From (1) we conclude that \( \Pi \) is the semidirect product

\[
\Pi = \tilde{\Pi} \times_{\varphi} \mathbb{Z}
\]

where \( \varphi : \mathbb{Z} \to \text{Aut} \tilde{\Pi} \) can be described as follows: By (3),

\[
\tilde{\Pi} = \cdots *_{H} G^{(-1)} *_{H} G^{(0)} *_{H} G^{(1)} *_{H} \cdots \quad (G^{(j)} \approx G, j \in \mathbb{Z})
\]

let \( \varphi(r) : \tilde{\Pi} \to \tilde{\Pi} \) be the automorphism defined by \( g_j \mapsto g_{j+r} \) for \( g_j \in G^{(j)} \).

Thus \( \Pi \) can be thought of as the group of the \( (g, t^j) \) \( (g \in \tilde{\Pi}, j \in \mathbb{Z}) \) with multiplication \( (g, t^j)(h, t^k) = (\varphi(s)(g) \cdot h, t^{j+k}) \).

Let \( \Gamma \) (resp. \( \tilde{\Gamma} \)) be the integral group ring of \( \Pi \) (resp. \( \tilde{\Pi} \)); then

\[
\Gamma = \tilde{\Gamma} \times_{\varphi} \mathbb{L}
\]

where \( \times_{\varphi} \) indicates twisted tensor product [20]. \( \Gamma \) can be described as \( \tilde{\Gamma} \otimes_{\mathbb{L}} \) with relations \( g_j \otimes t^r = g_{j+r} \otimes 1 \) for \( g_j \in G^{(j)} \) and \( r \in \mathbb{Z} \).

With the notation of §0, the inclusion \( Y(0) \subset X \) induces homomorphisms \( G \to \tilde{\Pi} \) and \( \pi_2(Y) \to \pi_2(X) \). The first one defines a structure of \( G \)-module on \( \tilde{\Pi} \) and, by (12'), on \( \Gamma \). Recall \( \pi_2(Y) \) is a \( G \)-module (cf. [4, p. 125]) which allows us to construct \( \pi_2(Y) \otimes_{G} \Gamma \). The induced map of second homotopy induces in turn a homomorphism \( e : \pi_2(Y) \otimes_{G} \Gamma \to \pi_2(X) \).

5. The elements of \( \pi_2(X) \). Suppose \( k : S^n \to S^{n+2} \) is a differentiable knot which admits a minimal Seifert manifold \( V \) in the sense of Lemma 1.

Lemma 5. The map \( e : \pi_2(Y) \otimes_{G} \Gamma \to \pi_2(X) \), defined by inclusion \( Y(0) \subset X \), is an epimorphism of \( \Gamma \)-modules.
Proof. We can represent $\sigma \in \pi_2(\tilde{X})$ by a map $f: S^2 \to \tilde{X}$ which is an embedding if $n \geq 3$ or whose self-intersection points do not belong to $\tilde{V}_j = V_0(j+1) = V_1(j)$ in the notation of [18, §0]. The $\tilde{V}_j$ are the liftings to $\tilde{X}$ of the Seifert manifold $V$ [7]. Let $\tilde{F} = \bigcup \tilde{V}_j$ and $M = f^{-1}(\tilde{F})$ a union of circles in $S^2$ and let $C$ be an innermost [6] component of $M$; $f(C)$ bounds a disk $f(D)$ in, say, $Y(j_0)$ and lies in $V_j(j_0)$ for $t = 0$ or 1. Then $C$ determines an element in $\ker \nu$. Since $V$ is a minimal manifold, $C$ bounds a disk $D'$ in $V_j(j_0)$. We can “pinch” $\sigma$ along $D'$ to obtain a new representative of $\sigma$ of the form $\sigma' + \gamma \sigma''$ where $\sigma'' \in \pi_2(Y)$ and $\gamma \in \tilde{\Pi}$. The element $\sigma'$ can be represented by a map $f': S^2 \to \tilde{X}$ and $f'^{-1}(\tilde{F})$ has one less innermost component. By our definitions and by induction, the proof is now complete.

6. Relations in $\pi_2(X)$. We now want to find $K = \ker(e)$. Let $x = \sum_i \sigma_i \otimes \gamma_i \in \pi_2(Y) \otimes \Gamma$ be in $K$; we can find a map $f: D^3 \to \tilde{X}$ such that the pinching process of §5 applied on $f|D^3$ yields $x$. For $n \geq 5$, $f$ can be taken to be an embedding; for $n = 2, 3, 4$ the self-intersections can be made transversal to $\tilde{V}$. Let $N = f^{-1}(\tilde{V})$ a disjoint union of orientable surfaces $F_j$ with (circular) boundary, properly embedded in $D^3$. Suppose $\partial F_j \subset \partial D^3$ is innermost (i.e., it bounds a 2-disk $\Delta$ in $\partial D^3$); let $\sigma \in \pi_1(F_j)$ be a generator of the fundamental group. Then, since $\nu \sigma = 0$ for some $t$, the loop $\sigma \subset \tilde{V}_j$ bounds a disk $\Delta_1$ in $Y(j_0) \cap f(D^3)$ and, since $V$ is minimal, a disk $\Delta_0$ in $\tilde{V}_j$. A homotopy that pulls $\Delta_1$ down to $\tilde{V}_j$ can be used to drag $f(D^3)$ along. This alters $f$ in such a way that a surgery is done to $F_j$ along $\sigma$. After finitely many steps $F_j$ becomes a disk $\Delta_2$. The sphere $f(\Delta_1) \cup f(\Delta_2) \subset Y(j_0)$ (identified along $f(\partial F_j)$) represents an element $x'$ of $\pi_2(Y(j_0))$ which deforms to $\sigma \in \pi_2(V_j)$. Thus $x$ can be deformed to an element of the form $x' + x''$, where $x'$ has been defined above and where $x''$ can be described by a map $f'' : D^3 \to X$ with image disjoint from $Y(j_0)$. We can pull $x'$ down to $Y(j_0 - 1)$ by noticing that $x' - v_1(\sigma) = 0$ in $\pi_2(Y(j_0))$ and that $v_0(\sigma) \otimes t = v_1(\sigma) \otimes 1$; so,

$$x = x'' + (v_0(\sigma) \otimes 1 - v_1(\sigma) \otimes 1).$$

Define $\pi_2(V) \otimes_H \Gamma$ by noticing that $\pi_2(V)$ is an $H$-module and that $v_0: H \to \Gamma$ defines an $H$-module structure on $\Gamma$ by means of the formula $h \gamma = v_0(h) \gamma$. If we use $v_1$ instead, we get $tv_0(\gamma) = v_1(h) \gamma$. Define now $d: \pi_2(V) \otimes_H \Gamma \to \pi_2(Y) \otimes_G \Gamma$ by $d(\sigma \otimes \gamma) = v_0(\sigma) \otimes t \gamma - v_1(\sigma) \otimes \gamma$, a homomorphism of $\Gamma$-modules.

Theorem 6. The sequence

$$\pi_2(V) \otimes_H \Gamma \xrightarrow{d} \pi_2(Y) \otimes_G \Gamma \xrightarrow{e} \pi_2(X) \to 0$$

is an exact sequence of $\Gamma$-modules.
In fact, by the discussion above, if $x \in K$, $x$ is a $\Gamma$-linear combination of elements of the form $d(\sigma \otimes 1)$.

7. Examples. I. Let $k: S^1 \to S^3$ be a knot and $F$ a minimal Seifert surface (cf. [9, Chapter IV]). Let $K: S^2 \to S^4$ be the 2-knot obtained by spinning [8]. The 3-manifold $V$ obtained from $F$ by spinning is a minimal Seifert manifold for $K$. Similarly, $Y$ obtained from $S^4$ by cutting along $V$ is obtained from $S^3 - F = Y_0$ by spinning. Since $Y_0$ is aspherical [19], it is clear that $\pi_2(Y)$ is a free $G$-module (cf. [16, p. 216]). Similarly $\pi_2(V)$ is a free $H$-module. The rank of both is twice the genus $g$ of $k$. The $\Gamma$-modules $\pi_2(V) \otimes_H \Gamma$ and $\pi_2(Y) \otimes_G \Gamma$ are then free $\Gamma$-modules of rank $2g$. Let $a_1, \ldots, a_{2g}$ (resp. $b_1, \ldots, b_{2g}$) be generators of $\pi_2(V)$ (resp. of $G$); the $a_i$ are in a one-to-one correspondence with the generators of $\pi_1(F)$. By (12), the fundamental group of the knot is presented by

$$\langle t, a_1, \ldots, a_{2g} \mid R_i = 1, i = 1, \ldots, 2g \rangle$$

where $R_i = t a_0(a_i) t^{-1} v_i(a_i)^{-1}$. The matrix of the map $d$ defined in §6 is then the Jacobian [17, p. 125] of (13), i.e., the matrix $\| \partial R_i / \partial b_j \|$. This follows at once from the discussion on the Jacobian in [9, p. 42]. This is a new proof of Theorem 2 of [8].

II. As in [12, II], let $(B^5, B^3)$ be the unknotted ball pair, $\partial B^3$ bounds a disk $\Delta$ in $\partial B^5$. Attach a 1-handle $h^1$ to $\partial B^5 - \Delta$. Let $L = (B^5 - B^3) \cup h^1$; $\partial L$ has the homotopy type of $S^1 \vee S^1 \vee S^3$. Let $\alpha$ be the generator of $\pi_1(\partial L)$ which goes around the handle and $\beta$ the meridian of $\partial B^3 \subset \partial B^5$. The loop $\beta$ pierces $\Delta$ at one point. Attach a 2-handle $h^2$ along $\alpha^2 \beta \alpha^{-1} \beta^{-1}$. We know $\partial (L \cup h^2)$ is the complement of a (slice) knot $k: S^2 \to S^4$ and a minimal Seifert manifold for $k$ can be obtained from $\Delta$ by attaching a tube so that $\Delta \cup (\text{tube})$ misses $\alpha^2 \beta \alpha^{-1} \beta^{-1}$. The Seifert manifold is thus obtained from $S^1 \times S^3$ by punching a hole to it. Thus $H = \mathbb{Z} \sigma$ and $G = \mathbb{Z} \alpha$; $\nu_i$ is given by multiplication by $t + 1$ ($t = 0, 1$). On the other hand $\pi_2(V) = \mathbb{Z} [H]$ and $\pi_2(Y) = \mathbb{Z} [G]$. We can choose $\nu_0(1) = 1$; then, by the argument in [12, p. 234], $2 \nu_0(1) - \alpha^{-1} \nu_1(1) = 0$. By Theorem 6, the sequence

$$\Gamma \xrightarrow{d} \Gamma \xrightarrow{\pi_2(X)} 0$$

where $d(1) = \beta - 2 \alpha$, is an exact sequence of $\Gamma$-modules. Then $\pi_2(X) = \mathbb{Z}_{(2)}$, where $\alpha$ acts trivially and $\beta^{-1}$ acts by multiplication by two (cf. [2, p. 130]).

III. In the above example, both $\pi_2(V)_H$ and $\pi_2(Y)_G$ are infinite cyclic and $\nu_i$ is given by multiplication by $t + 1$; similarly, if $\Pi$ is a knot group with deficiency one, the construction of [12] allows us to find a knot with minimal Seifert manifold obtained by adding 1-handles to a 3-disk. Then
both $\pi_2(V)_H$ and $\pi_2(Y)_G$ are free abelian and the map $\nu_0 - \nu_1$ is unimodular. By Lemma 3, $H_3(\mathbb{I}) = 0$.

IV. Observe that the sequence of Theorem 6 yields, by cancellation of the actions of the groups $H$ and $G$, the sequence (2) of Theorem 0. In example III, since $\alpha$ acts trivially, $\pi_2(X) = \pi_2(X)_{\mathbb{I}}$.

The author wishes to thank the referee for his valuable suggestions that led to a significant simplification of the statement of Theorem 6.

REFERENCES