THE EQUATION $L(E, X^{**}) = L(E, X)^{**}$
AND THE PRINCIPLE OF LOCAL REFLEXIVITY

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Abstract. A new derivation of the equation $L(E, X^{**}) = L(E, X)^{**}$ is given, for dim$(E) < \infty$ and $X$ a Banach space. From this equation the principle of local reflexivity is derived.

0. Introduction. The principle of local reflexivity [6] in the somewhat stronger form found in [5] is derived here from the equation $L(E, X)^{**} = L(E, X^{**})$. This equation is found explicitly in Schatten's monograph [7, pp. 40, 41] and at least implicitly in [3, p. 13]. Thus it is a classic formula in the theory of tensor products of Banach spaces. In §3 we use nontensor product methods to derive the equation $L(E, X^{**}) = L(E, X)^{**}$. The argument is easily accessible to students in a first functional analysis course.

1. Notation and preliminaries. Always $X$, $Y$, $Z$ are Banach spaces and $A$, $E$, $F$ are finite dimensional Banach spaces. All operators $S$, $T$, $U$, $V$, $W$ are continuous linear operators and the Banach space of operators from $X$ to $Y$ is denoted by $L(X, Y)$. Always $X$ is identified with its natural embedding in $X^{**}$. If $T \in L(X, Y^{**})$ and $(T_a) \subset L(X, Y)$ is a net such that $\lim_{a} f(T_{ax}) = T(x)$ for each $f \in Y^{*}$, $x \in X$ write $\omega^{*-}\operatorname{lim} T_a = T$. In the weak-star operator topology.

Lemma 1. Let $(T_a)$ be a net in $L(X, Y)$ and $T \in L(X, Y^{**})$ with $\|T_a\| \leq \|T\|$ for each $a$. Suppose $A \subset X$, $TA \subset Y$ and $\omega^{*-}\operatorname{lim} a T_a = T$. Then, to $\varepsilon > 0$, there is a net $(S_a) \subset L(X, Y)$ such that $\|S_a\| < \|T\| + \varepsilon$, $\omega^{*-}\operatorname{lim} S_a = T$ and $S_a a = Ta$ for each $a$ in $A$.

Proof. For each $a$ in $A$, $(T_a a)$ converges weakly to $Ta$. Since dim$(A) < \infty$, using standard techniques (e.g. [2, p. 477]), a net of convex combinations of $(T_a)$, say $(U_a)$, converges in norm on $A$, and $\omega^{*-}\operatorname{lim} U_a = T$. Write $X = A \oplus Z$ and set $S_a(a + f) = Ta + U_a f$. Then $S_a \in L(X, Y)$ and $\|S_a - U_a\| \leq \varepsilon$. Thus, for large $a$, $\|S_a\| < \|T\| + \varepsilon$.

If dim$(E) = 1$, then $L(E, X)^{**} = L(E, X^{**})$ is simply the statement that the unit ball $U_1(X) = \{x, \|x\| \leq 1\}$ is weak-star dense in $U_1(X^{**})$. As
described below, the equation means that, for each $T$ in $L(E, X^{**})$, there is a net $(T_n) \subseteq L(E, X)$ such that $\|T_n\| \leq \|T\|$ and $w^*\text{-}\lim T_n = T$.

To see that this is the meaning of the equation let $e_1, \ldots, e_n$ be a basis for $E$ and identify each $T$ in $L(E, X)$ with the usual coordinatewise vector and scalar operations and set $\|F\| = \sup\{\|\sum a_i x_i\|, \|\sum a_i e_i\| \leq 1\}$, so that the identification is an isometry between $L(E, X)$ and $Y$. Then $Y^* = \prod_{i}^n X^*$ with

$$\|x_i^*\| = \sup \left\{ \sum x_i^* (x_i) \left| \|x_i^*\| \leq 1 \right\}$$

and $Y^{**} = \prod_{i}^n X^{**}$ with $\|x_i^{**}\| = \sup\{\|\sum x_i^{**} (x_i^*)\|, \|x_i^{**}\| \leq 1\}$. Now associate each element $(x_i^{**})$ of $\prod_{i}^n X^{**}$ with the operator $T$ such that $T e_i = x_i^{**}$. If $\|x_i^{**}\| = 1$ and $\|(x_i^*)\| \leq 1$ such that $\sum x_i^* (x_i^*) \rightarrow \sum x_i^{**} (x_i^*)$ for each $(x_i^*)$ in $X^*$, then $x^* (\sum b_i x_i^*) \rightarrow (\sum b_i x_i^{**})(x^*)$ for each $x^*$ in $X^*$. Thus $\|\sum b_i x_i^{**}\| \leq \sum b_i \|x_i^{**}\|$. It easily follows that, if $\sqrt{1} > 0$, $\|x_i^*\| = \sup\{\|\sum b_i x_i^{**}\|, \|\sum b_i e_i\| \leq 1\} \geq (1 - \sqrt{1}) \|T\|$ for large $\alpha$ or $\|x_i^{**}\| \geq \|T\|$.

In summary, the mapping $(x_i^{**}) \rightarrow T$ is a norm decreasing mapping from $L(E, X^{**})$ onto $L(E, X^{**})$ which is continuous with the weak-star topology on $L(E, X^{**})$ and the weak-star operator topology on $L(E, X^{**})$. Further it is the identity on $L(E, X)$. The equation $L(E, X^{**}) = L(E, X^{**})$ means this mapping is an isometry.

2. Local reflexivity. Let $F$ be subspace of $X^*$ with basis $\{f_1, \ldots, f_k\}$ and let $T \in L(E, X^{**})$ with $E$ having basis $\{e_1, \ldots, e_n\}$ such that $[e_1, \ldots, e_m] = E \cap X$. The pairs $(f_i, e_j)$ define functionals on $L(E, X)$ by $(f_i, e_j)(S) = f_i (S e_j)$ (it is easy to compute that $\|(f_i, e_j)\| = \|f_i\| \|e_j\|$). Using Helly’s theorem (e.g. [8, p. 103]), and the equation $L(E, X^{**}) = L(E, X^{**})$ there is an $S$ such that $\|S\| = 1 + \sqrt{1}$ and $(f_i, e_j)(S) = T e_i (f_j)$ for each $i, j$. Thus $f (S e) = T e (f)$ for every $f$ in $F$, $e$ in $E$. (This argument is used in Lemma 1, [4].) One may assume, by enlarging $F$ if necessary, that for each $e$ there is a norm one $f$ such that $(1 - \sqrt{1}) \|Te\| < \|Te\|$.

**Theorem 1 (local reflexivity).** Let $E \subseteq X^{**}$, $A = E \cap X$, and $F \subseteq X^*$. To $\delta > 0$, there is an $S$ in $L(E, X)$ such that $(1 - \sqrt{1}) \|e\| < \|Se\| < (1 + \sqrt{1}) \|e\|$, $S a = a$ for each $a$ in $A$, and $f (S e) = e (f)$ for each $e$ in $E$, $f$ in $F$.

**Proof.** As in the calculation preceding Theorem 1, enlarging $F$ if necessary, assume $(1 - \sqrt{1}) \|e\| < \sup\{e (f)\} \|f\| \leq 1$, $f$ in $F$. Letting $T$ be the identity operator from $E$ to $X^{**}$ construct $(T_n)$ such that $(1 - \sqrt{1}) \|e\| < \sup\{e (f)\} \|f\| \leq 1$, $f$ in $F$. Letting $T$ be the identity operator from $E$ to $X^{**}$ construct $(T_n)$ such that $(1 - \sqrt{1}) \|e\| < \sup\{e (f)\} \|f\| \leq 1$,
\[ \| T_e e \| < (1 + \delta) \text{ if } \| e \| = 1. \] Then \((1 - \delta/2)\| e \| < \| T_e e \| < \| e \|(1 + \delta) \) for every \( e \) and set \( S = T_e \) for some \( \alpha \).

3. **The derivation of** \( L(E, X^{**}) = L(E, X)** \). If \( E = l_{1,n} \) then the derivation is as follows. Let \( e_1, \ldots, e_n \) be the usual unit vector basis of \( l_{1,n} = E \). For \( T \) in \( L(E, X) \), \( \| T \| = \text{sup} \{ \| \sum \alpha_i T e_i \|, \sum |\alpha_i| \leq 1 \} \leq \max \{ \| T e_i \| \} \). But \( \| T \| \geq \max \| T e_i \| \) since \( \| e_i \| = 1 \) for each \( i \). Thus \( Y = \prod_1^n X \) has norm \( \| (x_i) \| = \max \{ \| x_i \| \} \). Then \( Y^* = \prod_1^n X^* \) has norm, \( \| (x_i^*) \| = \sum |x_i^*| \) and \( Y^{**} = \prod_1^n X^{**} \) has norm, \( \| (x_i^{**}) \| = \max \{ \| x_i^{**} \| \} \). The latter is the norm for \( L(E, X^{**}) \) so that the mapping of \( Y^{**} \) to \( L(E, X^{**}) \) in §1 is an isometry.

Now let \( E, \varepsilon > 0 \) be given and let \( V \) be an operator on \( l_{1,n} \) to \( E \) such that \( V \{ (u| |u| < 1 + \varepsilon) \} = \{ e| \| e \| \leq 1 \} \). That such \( l_{1,n}, V \) exist may be seen by embedding \( E^* \) into an \( l_{\infty,k} \) in such a way that \( \| e^* \| \geq \| U e^* \| \geq (1 - \alpha) \| e^* \| \) and choosing \( \alpha \) small and \( V = U^* \). If \( T \in L(E, X^{**}) \), then \( T V \in L(l_{1,n}, X^{**}) \) and \( \| T V \| \leq \| T \| \). Set \( A = \{ u| TV u \in X \} \). There is a net \( (S_{\alpha}) \) in \( L(l_{1,n}, X) \) such that \( \| S_{\alpha} \| \leq \| TV \| (1 + \varepsilon) \), \( \varepsilon \)-op \( \lim S_{\alpha} = TV \), and by Lemma 1 we find \( S_{\alpha} \) such that \( S_{\alpha} u = TV u \) if \( u \in A \). In particular if \( V u = 0 \) then \( S_{\alpha} u = 0 \). Define \( T_{\alpha} \in L(E, X) \) by letting \( T_{\alpha} e = S_{\alpha} u \) if \( V u = e \). Because \( V u = 0 \) implies \( S_{\alpha} u = 0 \) one has that \( T_{\alpha} \) is well defined and in \( L(E, X) \). Moreover \( T_{\alpha} V = S_{\alpha} \). If \( \| e \| \leq 1 \) and \( \| u \| < 1 + \varepsilon \) such that \( V u = e \), then \( \| T_{\alpha} e \| = \| S_{\alpha} u \| \leq \| S_{\alpha} \| (1 + \varepsilon) \leq \| TV \| (1 + \varepsilon) \leq \| T \| (1 + \varepsilon)^2 \). Finally \( x^*(T_{\alpha} V u) \rightarrow (TV u) x^* \) and so \( x^*(T_{\alpha} e) \rightarrow (T e)(x^*) \) for every \( e \) in \( X^* \). Thus \( \varepsilon \)-op \( \lim T_{\alpha} = T \) since \( \varepsilon > 0 \) is arbitrary the mapping from \( L(E, X^{**}) \) to \( L(E, X^{**}) \) at the end of §1 is an isometry. This concludes the derivation.

**Bibliography**