THE EQUATION \( L(E, X^{**}) = L(E, X)^{**} \)
AND THE PRINCIPLE OF LOCAL REFLEXIVITY

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Abstract. A new derivation of the equation \( L(E, X^{**}) = L(E, X)^{**} \) is given, for \( \text{dim}(E) < \infty \) and \( X \) a Banach space. From this equation the principle of local reflexivity is derived.

0. Introduction. The principle of local reflexivity [6] in the somewhat stronger form found in [5] is derived here from the equation \( L(E, X)^{**} = L(E, X^{**}) \). This equation is found explicitly in Schatten's monograph [7, pp. 40, 41] and at least implicitly in [3, p. 13]. Thus it is a classic formula in the theory of tensor products of Banach spaces. In §3 we use nontensor product methods to derive the equation \( L(E, X^{**}) = L(E, X)^{**} \). The argument is easily accessible to students in a first functional analysis course.

1. Notation and preliminaries. Always \( X, Y, Z \) are Banach spaces and \( A, E, F \) are finite dimensional Banach spaces. All operators \( S, T, U, V, W \) are continuous linear operators and the Banach space of operators from \( X \) to \( Y \) is denoted by \( L(X, Y) \). Always \( X \) is identified with its natural embedding in \( X^{**} \). If \( T \in L(X, Y^{**}) \) and \( (T_a) \subset L(X, Y) \) is a net such that \( \lim_{a} T_{a}(x) = T(x) \) for each \( x \in X \) write \( w^{*}\text{-op} \lim T_a = T \) (\( T_a \to T \) in the weak-star operator topology).

Lemma 1. Let \( (T_a) \) be a net in \( L(X, Y) \) and \( T \in L(X, Y^{**}) \) with \( \|T_a\| \leq \|T\| \) for each \( a \). Suppose \( A \subset X \), \( TA \subset Y \) and \( w^{*}\text{-op} \lim_a T_a = T \). Then, to \( \epsilon > 0 \), there is a net \( (S_a) \subset L(X, Y) \) such that \( \|S_a\| < \|T\| + \epsilon \), \( w^{*}\text{-op} \lim S_a = T \), and \( S_a x = T_a x \) for each \( x \in A \).

Proof. For each \( a \in A \), \( (T_a x) \) converges weakly to \( T_a x \). Since \( \text{dim}(A) < \infty \), using standard techniques (e.g. [2, p. 477]), a net of convex combinations of \( (T_a) \), say \( (U_a) \), converges in norm on \( A \), and \( w^{*}\text{-lim} U_a = T \). Write \( X = A \oplus Z \) and set \( S_a(a + f) = T_a + U_a f \). Then \( S_a \in L(X, Y) \) and \( \|S_a - U_a\| \to 0 \). Thus, for large \( a \), \( \|S_a\| < \|T\| + \epsilon \).

If \( \text{dim}(E) = 1 \), then \( L(E, X)^{**} = L(E, X^{**}) \) is simply the statement that the unit ball \( U_1(X) = \{ x, \|x\| \leq 1 \} \) is weak-star dense in \( U_1(X^{**}) \). As
described below, the equation means that, for each \( T \) in \( L(E, X**) \), there is a net \((T_\alpha) \subset L(E, X)\) such that \( \|T_\alpha\| \leq \|T\| \) and \( w^*\)-op \( \lim \alpha \) \( T_\alpha = T \). To see that this is the meaning of the equation let \( e_1, \ldots, e_n \) be a basis for \( E \) and identify each \( T \) in \( L(E, X) \) with the usual coordinatewise vector and scalar operations and set \( \|x\| = \sup\{|\sum a_i x_i|, \sum |a_i| \leq 1\} \), so that the identification is an isometry between \( L(E, X) \) and \( Y \). Then \( Y^* = \prod^n X^* \) with

\[
\|x^*_i\| = \sup \left\{ \sum x^*_i (x_i) \mid \|x_i\| < 1 \right\}
\]

and \( Y** = \prod^n X^{**} \) with \( \|x^{**}_i\| = \sup \{|\sum x^{**}_i (x^*_i)| \|x^*_i\| \leq 1\} \). Now associate each element \((x^{**}_i)\) of \( \prod^n X^{**} \) with the operator \( T \) such that \( T e_i = x^{**}_i \). If \( \|x^{**}_i\| = 1 \) and \( \|x^*_i\| \leq 1 \) such that \( \sum x^*_i (x^*_i) \rightarrow \sum x^{**}_i (x^{**}_i) \) for each \((x^*_i)\) in \( X^* \), then \( x^*_i (\sum b_i x^*_i) \rightarrow (\sum b_i x^{**}_i)(x^*) \) for each \( x^* \) in \( X^* \). Thus \( \lim \|\sum b_i x^*_i\| \|T\| \leq \|\sum b_i x^{**}_i\| \) \( \|T\| \) for \( \alpha \). It easily follows that, if \( \varepsilon > 0 \), \( \|x^*_i\| = \sup\{|\sum b_i x^*_i| \|x^*_i\| \leq 1\} \) \( \|T\| \) for large \( \alpha \) or \( \|x^{**}_i\| \leq \|T\| \).

In summary, the mapping \((x^{**}_i) \rightarrow T\) is a norm decreasing mapping from \( L(E, X)^{**} \) onto \( L(E, X^{**}) \) which is continuous with the weak-star topology on \( L(E, X)^{**} \) and the weak-star operator topology on \( L(E, X^{**}) \). Further it is the identity on \( L(E, X) \). The equation \( L(E, X^{**}) = L(E, X)^{**} \) means this mapping is an isometry.

2. Local reflexivity. Let \( F \) be subspace of \( X^* \) with basis \( \{f_1, \ldots, f_k\} \) and let \( T \in L(E, X^{**}) \) with \( E \) having basis \( \{e_1, \ldots, e_n\} \) such that \( [e_1, \ldots, e_m] = E \cap X \). The pairs \((f_i, e_j)\) define functionals on \( L(E, X)\) by \((f_i, e_j)(S) = f_i(S e_j)\) (it is easy to compute that \( \|f_i\| = \|f_i\| \|e_j\| \)). Using Helly's theorem (e.g. [8, p. 103]), and the equation \( L(E, X)^{**} = L(E, X^{**}) \) there is an \( S \) such that \( \|S\| < \|T\| + \varepsilon \) and \((f_i, e_j)(S) = T e_j(f_i)\) for each \( i, j \). Thus \( f(S e^*) = T e^*(f) \) for every \( f \in F, e \in E \). (This argument is used in Lemma 1, [4].) One may assume, by enlarging \( F \) if necessary, that for each \( e \) there is a norm one \( f \) in \( F \) such that \( (1 - \varepsilon)\|T\| < \|T e^*(f)\| \). Constructing \( S = S_G \) for each \( G \supset F \) such that \( g(S e^*) = T e^*(g) \) for each \( g \in G \), \( e \in E \), and such that \( \|S_G\| < \|T\| (1 + \varepsilon) \), then \( w^*\)-op \( \lim G S_G = T \). By Lemma 1 there is a net \((T_\alpha) \subset L(E, X)\) such that \( w^*\)-op \( \lim T_\alpha = T \) and \( T_\alpha e_i = e_i \) if \( i \leq m, f(T_\alpha e) = T e^*(f) \) for each \( e \) in \( E, f \) in \( F \), and \( \|T_\alpha\| < \|T\| (1 + 2\varepsilon) \). Further, \((1 - \varepsilon)\|T\| < \|T_\alpha e\| < \|T\| (1 + 2\varepsilon) \) if \( \|e\| \leq 1 \).

**Theorem 1 (Local Reflexivity).** Let \( E \subset X^{**}, A = E \cap X \), and \( F \subset X^* \). To \( \delta > 0 \), there is an \( S \) in \( L(E, X) \) such that \( (1 - \delta)\|e\| < \|S e\| < (1 + \delta)\|e\| \), \( S a = a \) for each \( a \) in \( A \), and \( f(S e) = e^*(f) \) for each \( e \) in \( E, f \) in \( F \).

**Proof.** As in the calculation preceding Theorem 1, enlarging \( F \) if necessary, assume \((1 - \delta/2)\|e\| < \sup\{e(f)\| f \| \leq 1, f \in F\} \). Letting \( T \) be the identity operator from \( E \) to \( X^{**} \) construct \((T_\alpha)\) such that \((1 - \delta/2) <
\[ \|T_xe\| < (1 + \delta) \] if \[ \|e\| = 1. \] Then \[ (1 - \delta/2)\|e\| < \|T_xe\| < \|e\|(1 + \delta) \] for every \( e \) and set \( S = T_x \) for some \( \alpha \).

3. The derivation of \( L(E, X^{**}) = L(E, X)^{**} \). If \( E = l_{1,n} \) then the derivation is as follows. Let \( \{e_1, \ldots, e_n\} \) be the usual unit vector basis of \( l_{1,n} = E \). For \( T \) in \( L(E, X) \), \[ \|T\| = \sup \{\|\sum x_iTe_i\|, \sum |x_i| \leq 1\} \leq \max \{\|Te_i\|\} \]. But \[ \|T\| \geq \max \{\|Te_i\| \} \] since \( |e_i| = 1 \) for each \( i \). Thus \( Y = \sum_1^n X \) has norm \( \|(x_i)\| = \max \{\|x_i\|\} \). Then \( Y^* = \sum_1^n X^* \) has norm, \( \|(x_i^*)\| = \sum |x_i^*| \) and \( Y^{**} = \sum_1^n X^{**} \) has norm, \( \|(x_i^*)\| = \max \{\|x_i^*\|\} \). The latter is the norm for \( L(E, X^{**}) \) so that the mapping of \( Y^{**} \) to \( L(E, X^{**}) \) in §1 is an isometry.

Now let \( E, \epsilon > 0 \) be given and let \( V \) be an operator on \( l_{1,n} \) to \( E \) such that \( V(\{u| |u| < 1 + \epsilon\}) = \{e| |e| \leq 1\} \). That such \( l_{1,n} \), \( V \) exist may be seen by embedding \( E^* \) into an \( l_{1,n} \) in such a way that \( \|\epsilon^*\| \geq \|Ue^*\| \geq (1 - \alpha)\|e^*\| \) and choosing \( \alpha \) small and \( V = U^* \). If \( T \in L(E, X^{**}) \), then \( TV \in L(l_{1,n}, X^{**}) \) and \( \|TV\| \leq \|T\| \). Set \( A = \{u| TVu \in X\} \) and find \( S_a \) such that \( S_a \subseteq TV \), \( w^*-\lim S_a = TV \), and by Lemma 1 we find \( S_a \) such that \( S_a u = TVu \) if \( u \in A \). In particular if \( Vu = 0 \) then \( S_a u = 0 \). Define \( T_a \in L(E, X) \) by letting \( T_a e = S_a u \) if \( Vu = e \). Because \( Vu = 0 \) implies \( S_a u = 0 \) one has that \( T_a \) is well defined and in \( L(E, X) \). Moreover \( T_a V = S_a \). If \( |e| \leq 1 \) and \( |u| < 1 + \epsilon \) such that \( Vu = e \), then \( \|T_a e\| = \|S_a u\| \leq \|S_a\|(1 + \epsilon) \leq \|TV\|(1 + \epsilon)^2 \leq \|T\|(1 + \epsilon)^2 \). Finally \( x^*(T_a Vu) \rightarrow (TVu)x^* \) and so \( x^*(T_a e) \rightarrow (Te)(x^*) \) for every \( e \) in \( X^* \). Thus \( w^*-\lim T_a = T \). Since \( \epsilon > 0 \) is arbitrary the mapping from \( L(E, X)^{**} \) to \( L(E, X^{**}) \) at the end of §1 is an isometry. This concludes the derivation.

**Bibliography**


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