THE COBORDISM OF INVOLUTIONS
ON ORIENTABLE MANIFOLDS

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Abstract. This note calculates the cobordism of smooth, oriented manifolds with orientation-reversing involution, and also the 2-torsion in the cobordism of such manifolds with orientation-preserving involution. In particular, it is shown that the forgetful homomorphism maps both groups monomorphically into the cobordism of unoriented manifolds with involution.

1. Statement of results. Let $O_*$ be the cobordism theory of smooth oriented manifolds $M$ with involution $T$; we do not require $T$ to be orientation preserving. Then $O_* = O_*^- \oplus O_*^+$, where the equivalence class $[M, T]$ is in $O_*^-$ (respectively, $O_*^+$), if $T$ is orientation reversing (respectively, preserving). The notation agrees with that of [4].

In this note we determine $O_*^\oplus \otimes \text{torsion } O_*^\oplus$. Rosenzweig [5] proved that the torsion of $O_*^\oplus$ is all of order 2. In $O_*^\oplus$ every class is of order 2, for if $T$ is orientation reversing,

$$(M \cup M, T \cup T) \cong \partial(M \times I, T \times \text{Id})$$

under the diffeomorphism $\text{Id} \cup T$.

$O_*^\oplus$ was also studied by Conner [2], and $O_*$ has recently been studied by Lee and Wasserman [4], who show how classes of $O_*$ are detected by equivariant characteristic numbers. Our results complement these two papers, and we offer a short proof (Lemma 4) of a result which appears in both.

Theorem 1. Let $I_*$ be the cobordism of unoriented manifolds with involution [3, Chapter IV]. The forgetful map $\rho: O_* \rightarrow I_*$ is monic on 2-torsion.

From [6] it follows that the torsion is mapped monomorphically into the Wall bordism group $\tilde{W}_* (\bigvee_{j=1}^{\infty} TBO(j))$ (see [7, Chapter VIII]). An element of this group is represented by a vector bundle $v \rightarrow F$, whose disk bundle $Dv$ is a Wall manifold [8], together with a classifying map

Received by the editors November 14, 1972.

AMS (MOS) subject classifications (1970). Primary 57D85; Secondary 57D85.

Key words and phrases. Equivariant cobordism, orientation-preserving involution, orientation-reversing involution, Wall manifold.
(Dv, Sν) → (√TBO(j), ∞). Define \( \partial'[Dv] = [Di^*ν] \), where \( i : N \subset F \) includes a submanifold dual to \( w_1F + w_1ν \). Then \( Di^*ν \) is dual to \( w_1Dv \), hence is oriented.

**Theorem 2.** \( \mathcal{O}^+_* \cong \text{Im} \partial' \cap \tilde{W}_*(\sqrt{\bigcup_{j=1}^∞ TBO(2j+1)}) \).

Let \( p : \tilde{W}_*(\sqrt{TBO(2j)}) \to W_* \) map \([Dv]\) to the class of the projective space bundle \( Pν \), and \( q : \tilde{W}_*(TBO(1)) \to \tilde{W}_*(\sqrt{TBO(2j)}) \) map \([Dv]\) to \([D(ν\oplus 1)]\). Then \( pq = 0 \) and \( \partial'(\text{Im} q) \subseteq \text{Im} q \), so \( p \) and \( \partial' \) induce

\[
p' : \tilde{W}_*(\sqrt{TBO(2j)})/\text{Im} q \to W_*
\]

and

\[
\partial'' : \tilde{W}_*(\sqrt{TBO(2j)})/\text{Im} q \to \tilde{W}_*(\sqrt{TBO(2j)})/\text{Im} q.
\]

**Theorem 3.** torsion \( \mathcal{O}^*_+ \cong \text{Im} \partial'' \cap \text{Ker} \rho' \).

These theorems are used to describe \( Z_2 \)-generators of \( \mathcal{O}^-_* \) and torsion \( \mathcal{O}^+_* \). The usefulness of the latter description is limited by lack of knowledge of \( \text{Ker} \rho' \).

I wish to thank Professors Lee and Wasserman for supplying an advance copy of [4], and also Professor R. E. Stong for his advice and encouragement.

2. **Wall manifolds with involution.** A Wall manifold with involution, \((W, T, f)\), is an unoriented manifold \( W \) with an involution \( T \) and a map \( f : W \to S^1 \) such that \( fT = f \) and \( f^*(ν) = w_1W \), where \( ν \in H^1(S^1; Z_2) \) is the generator. To construct examples one uses:

**Lemma 1** (See [6, (2.5)]). If \( Dv \) is a Wall manifold (respectively, orientable) and \( \dim ν \) is even, then \( Pν \) is a Wall manifold (respectively, orientable).

Let \( W_*^I \) and \( W_*^K \) be the cobordism groups of Wall manifolds with involution and with fixed point free involution, respectively. Following are the chief results of [6]. The forgetful map \( W_*^I \to I_* \) is monic. Clearly \( \text{Im} \rho \subseteq W_*^I \). If \( d : W_*^I \to \mathcal{O}_* \) assigns to \([W, T, f]\) the class of \((N, T|N)\), where \( N \subset W \) is an invariant submanifold dual to \( w_1W \), then the sequence

\[
\mathcal{O}_* \xrightarrow{\rho} W_*^I \xrightarrow{d} \mathcal{O}_* \xrightarrow{2} \mathcal{O}_*
\]

is exact. Define \( d' = \rho d : W_*^I \to W_*^I \). The map \( s : W_*^I \to \tilde{W}_*(\sqrt{TBO(j)}) \), given by classifying the normal bundle to the fixed set, is monic. The map \( \partial : \tilde{W}_*(\sqrt{TBO(j)}) \to W_*^F \), given by \( \partial[Dv] = [Sv, \text{antipodal map}] \), is epic.

We give a new proof of the last fact by constructing a splitting map. Recalling [6, (3.1)], if \([M, f] \in \tilde{W}_*(RP^{∞})\) let \( π : \tilde{M} \to M \) be the double cover classified by \( f \). If \([M, f] \in N_*^1(S^1)\) let \( π : \tilde{M} \to M \) be the double cover corresponding to \( f^*ν + w_1M \). In either case \( \tilde{M} \) is a Wall manifold with
involution $\Delta$, the deck transformation reversing sheets of $\tilde{M}$. This defines monomorphisms
\[ k : W_\ast(RP^n) \to W_\ast^{F}, \quad j : N_\ast(S^1) \to W_\ast^{F} \]
with $k+j$ epic and $\text{Im} k \cap \text{Im} j \cong W_\ast$, identified as the set of classes $[W \times S^0, \text{Id} \times \Delta]$ for $[W] \in W_\ast$.

Now let $\lambda : M \to M$ be the line bundle corresponding to $\pi$. If $[M, f] \in N_\ast(S^1)$, $D\lambda$ is a Wall manifold. Define
\[ \delta[\tilde{M}, \Delta] = [D\lambda] \in \tilde{W}_\ast(\sqrt{\text{TBO}(n)}). \]
If $[M, f] \in W_\ast(RP^n)$, let $i : N \subset M$ include a submanifold defining the double cover. $D(i^*\lambda)$ is a Wall manifold; by Lemma 1 so is $P = P(i^*\lambda \oplus 1)$. Define
\[ \delta[\tilde{M}, \Delta] = [D(i^*\lambda \oplus 1)] + [D(1 \to M)]. \]

**Lemma 2.** $\exists \delta = \text{Id} : W_\ast \to W_\ast^{F}$. 

**Proof.** For $x \in \text{Im} j$ it is obvious that $\partial \delta(x) = x$. If $[M, f] \in W_\ast(RP^n)$ write $P = D(i^*\lambda) \cup D(i^*\lambda)$. The map $g : P \to RP^n$, classifying the canonical line bundle, may be written
\[ g = (f|N)p_0 \cup * \]
where $p_0$ is the projection of $D(i^*\lambda)$ and $*$ is homotopic to a trivial map (see the proof of [3, (26.1)]). Since $i^*\lambda$ is the normal bundle of $N$ in $M$ we may also assume $f$ is of the form (1). Now form $(H, h)$ from $(M \times I, f_{\pi M})$ and $(P \times I, g_{\pi P})$ by sewing copies of $(D(i^*\lambda) \times 1, (f|N)p_0)$. \[ \partial(H, h) = (M, f) \cup (P, g) \cup (M, *) \]
which proves $\partial \delta[\tilde{M}, \Delta] = [\tilde{M}, \Delta]$. \qed

3. **Proof of Theorem 1.** The statement of the theorem can be sharpened:

**Theorem 1'.** $\rho$ maps torsion $\Theta_\ast$ isomorphically onto $\text{Im} \partial'$. 

**Lemma 3.** Let $(M, T)$ be an orientable manifold with involution, and $(M, T) = \partial(Q, U)$ as unoriented manifolds with involution. If $Q$ is in fact orientable, then
\[ [M, T] = 2\beta + \omega[S^0, \Delta] \]
for some $\beta \in \Theta_\ast^+$ and $\omega \in \Omega_\ast$.

**Proof.** Give $Q$ an orientation and let $\partial Q = M'$ be its oriented boundary.
\[ [M, T] = [M, T] - [M', T]. \]

Of course $M' = M$ up to the sign of the orientation on each component. $2x = 0$ for $x \in \Theta_\ast$; also the involution $M \cup (-M) \to M \cup (-M)$, reversing components, bounds $(M \times [-1, 1], \text{Id} \times (-1))$. (2) then reduces to the desired form. \qed
Lemma 4. If $x \in \mathcal{O}_n$ and $\rho x = 0$ then either $x \in 2\mathcal{O}_n$ or else $n = 4i$ and
\[
x = 2\beta + \omega[S^0, \Delta]
\]
for some $\beta \in \mathcal{O}^+_{4i}$ and $\omega \in \Omega_{4i}/\text{torsion}$.

Note. This is [4, Proposition 21] and, essentially, [2, Theorem 5.8]. Following is a short proof relying on [6].

Proof. Write $x = y + z$, $y \in \mathcal{O}^+_n$, $z \in \mathcal{O}_n^-$. Since $s\rho y$ and $s\rho z$ lie in orthogonal summands we must have $\rho y = 0 = \rho z$.

Let $(M, T)$ represent either $y$ or $z$, and let $(W, S, f)$ be a Wall manifold with involution and boundary $(M, T, *)$. By [6, (4.1)] we may assume $f(M) = \{a\}$ and $f$ is transverse regular at $b \neq a$. If $b \in \text{Im } f$, cut $W$ along $N = f^{-1}b$, as in [8, §1], obtaining an oriented cobordism of $(M, T)$ with $(N \times S^0, (S|N) \times 1d)$. This bounds if $S$ is orientation reversing. If $b \notin \text{Im } f$, then $W$ is orientable and Lemma 3 applies. Finally, Rosenzweig showed [5] that $2\beta = 0$ if $n \neq 4i$ and that $\omega[S^0, \Delta] = 0$ if $2\omega = 0$. □

Suppose $x \in \mathcal{O}_n$, $\rho x = 0 = 2x$. If $x \in 2\mathcal{O}_n$ then $x = 0$, for the only torsion is of order 2. If $x$ has the form (3), then since $2x = 0$ we have $4\beta = 2\omega[S^0, \Delta]$. By the proof of [2, Corollary 5.5], index $\omega$ is even. Then [2, Theorem 4.5] proves $\omega[S^0, \Delta] \in 2\mathcal{O}_{4i}^+$. Thus $x = 0$ in this case also, and $\rho$ is monic on the torsion. The reader can easily show that $\rho(\text{torsion}) = \text{Im } d'$, completing Theorem 1'. □

4. Calculation of $\partial'$. To give meaning to Theorems 2 and 3, we compute the homomorphism $\partial'$. The structure of $\tilde{W}_* (\sqrt{TBO(n)})$ is determined by [3, §17] and [7, pp. 163–164].

Lemma 5. $\tilde{W}_* (\sqrt{TBO(n)})$ is the $W_*$-polynomial algebra on generators $[D\lambda_i]$, one of each dimension $i \geq 1$, with $\lambda_i$ a line bundle. If $i = 2m + 1$, we may take $\lambda_i$ to be the canonical line bundle over $\mathbb{R}P^{2m}$. If $i = 2m + 2$, we may take $\lambda_i$ to be the canonical line bundle over $P(\xi \oplus 2m \to S^1)$, where $\xi \to S^1$ is the nontrivial line bundle.

The proof is perfectly straightforward, and all the needed cohomology data is listed in [6, (4.1)].

Recall that $\partial'[Dv] = [D\mathcal{I}^*v]$, if $v \to F$ and $i : N \subset F$ includes a submanifold dual to $w_1 v + w_1 F$. By [7, pp. 169–172], $Dv$ is orientable if and only if $[Dv] \in \text{Im } \partial'$. For future reference, if $v$ has even fiber dimension then by [6, (2.4)] it is also true that $P(i^* v)$ is dual in $Pv$ to $w_1 Pv$. Then
\[
2[P(i^*v)] = 0 \in \Omega_*,
\]
by [8].
Lemma 6. For all \( m \), \( \partial'[\lambda_{2m+2}] = [\lambda_{2m+1}] \) and \( \partial'[\lambda_{2m+1}] = 0 \). If \( \partial_1: W_* \to W_* \) maps \( W \) to the submanifold dual to \( w, W \) [8], then

\[
\partial'(w\beta) = (\partial_1 w)\beta + w(\partial'\beta)
\]

for all \( w \in W_* \), \( \beta \in W_\ast(\sqrt{TBO(n)}) \).

Remarks. The first statement is immediate and the second may be proved with characteristic numbers, exactly like [8, Lemma 3]. From the proof of [6, (4.1)], we see that \( \partial' s = s \partial' \). Since \( s \) is a \( W_* \)-monomorphism, the lemma also proves that \( \partial'(w\beta) = (\partial_1 w)\beta + w(\partial'\beta) \) for \( w \in W_* \), \( \beta \in W_\ast \).

5. Proof of Theorems 2 and 3. If \( v \) has odd fiber dimension, then by [6, (3.3)] we have \( \partial[\lambda] = [D\lambda] \), where \( \lambda \to P(v) \) is the canonical line bundle. By Lemma 1, \( P(v \oplus 1) \) is a Wall manifold. Let \( t(v) \) be the involution on \( P(v \oplus 1) \) induced by the linear map \( 1 \times (-1) \) in each fiber of \( v \oplus 1 \), and define

\[
\sigma[Dv] = [P(v \oplus 1), t(v)] \in W_*^I.
\]

Then \( \sigma[Dv] + \delta \partial[Dv] = [Dv] \), just as in [3, §28]. It follows that \( \sigma[Dv] = 0 \) if and only if \( [Dv] = \delta \partial[Dv] \in W_*^I(TBO(1)) \). Also, since \( \delta s = 0 \), it follows that \( \sigma p(x) = p(x) \) for \( x \in \partial_*^+ \). Since \( P(v \oplus 1) \) is orientable if \( Dv \) is, and \( t(v) \) is then orientation reversing, Theorem 2 is proved.

If \( v \to F \) has even fiber dimension then we will assume \( [Dv] \in \text{Ker} p \). To justify this restriction, suppose \( [M, T] \in \partial_*^+ \) and \( v \) is the normal bundle to the fixed set of \( T \). Then \( P(v) \) bounds \( (M - \text{Int} Dv)/T \), so \( p s p[M, T] = 0 \).

Suppose, then, that \( [Dv] \in \text{Ker} p \). Since \( P(v) \) is a Wall manifold, Lemma 2 implies \( \delta[Dv] = [D (i^* \lambda \oplus 1)] \), where \( i: N \subseteq P(v) \) includes a submanifold dual to \( \lambda \). We define \( \sigma: \text{Ker} p \to W_*^I \). Let \( N = S(i^* \lambda) \), so \( \tilde{N} \to N \) is a double cover and \( \tilde{N} \) is imbedded in \( S(v) \) with trivial normal bundle. \( \tilde{N} \) is also imbedded in \( S(i^* \lambda \oplus 1) \) with trivial normal bundle, so we sew \( D(v) \) and \( D(i^* \lambda \oplus 1) \) along the copies of \( \tilde{N} \times I \) in the boundaries. The remaining boundary is then two copies of \( P(v) \), which bounds some \( Q \). Sewing on \( 2Q \) builds a Wall manifold \( K(v) \), which admits an involution \( t(v) \) that restricts to the antipodal involution on the disk bundles, and reverses the two copies of \( Q \). Thus the fixed set of \( t(v) \) is the disjoint union of \( F \) and \( N \), and the normal bundles are \( v \) and \( i^* \lambda \oplus 1 \), respectively.

Define \( \sigma[Dv] = [K(v), t(v)] \in W_*^I \); it is easy to see that this is a well-defined homomorphism. By construction,

\[
\sigma[Dv] + \delta \partial[D\sigma] = [Dv].
\]

Reasoning as before, \( \sigma[Dv] = 0 \) if and only if \( [Dv] = \delta \partial[Dv] \in \text{Im} q \), and \( \sigma p(x) = p(x) \) for \( x \in \partial_*^+ \).
Finally, suppose \( Dv \) is oriented. Then so are \( D(i^{*}\lambda\oplus 1) \) and \( P_{v} \). As noted in the last section, \( P_{v} \) represents a torsion class in \( \Omega_{\ast} \). Since \( \Omega_{\ast}\rightarrow W_{\ast} \) is monic on torsion [8] we may assume \( Q \) is oriented. Thus \( K(v) \) is oriented, and \( t(v) \) is orientation preserving. This completes the proof of Theorem 3.

**Remarks.** (1) We have shown that \( \mathcal{O}_{\ast}^{-} \) is the \( \mathbb{Z}_{2} \)-vector space on generators \([P(v\oplus 1), t(v)]\), for \([Dv]\in \text{Im }\partial^{r}, \dim v=2m+1, m>0\). The torsion of \( \mathcal{O}_{\ast}^{+} \) is a \( \mathbb{Z}_{2} \)-vector space on generators \([K(v), t(v)]\) for 
\[ [Dv] \in \text{Im }\partial^{r} \cap \text{Ker }p', \]
but this is not very helpful, since we cannot actually construct \( K(v) \) without knowing \( Q=P_{v} \).

(2) If \([M, T]\in \mathcal{O}_{\ast}^{-}, \) then clearly \([M]\) is a torsion class in \( \Omega_{\ast} \). Conversely, P. G. Anderson proved [1, Proposition 5] that every torsion class in \( \Omega_{\ast} \) is represented by one of our examples \( P(v\oplus 1). \)

**References**


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