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THE COBORDISM OF INVOLUTIONS ON ORIENTABLE MANIFOLDS

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ABSTRACT. This note calculates the cobordism of smooth, oriented manifolds with orientation-reversing involution, and also the 2-torsion in the cobordism of such manifolds with orientation-preserving involution. In particular, it is shown that the forgetful homomorphism maps both groups monomorphically into the cobordism of unoriented manifolds with involution.

1. **Statement of results.** Let $\mathcal{O}_*$ be the cobordism theory of smooth oriented manifolds M with involution T; we do not require T to be orientation preserving. Then $\mathcal{O}_* = \mathcal{O}_*^- \oplus \mathcal{O}_*^+$, where the equivalence class $[M, T]$ is in $\mathcal{O}_*^-$ (respectively, $\mathcal{O}_*^+$), if T is orientation reversing (respectively, preserving). The notation agrees with that of [**4**].

In this note we determine $\mathcal{O}_*^- \oplus \text{torsion}\, \mathcal{O}_*^+$. Rosenzweig [**5**] proved that the torsion of $\mathcal{O}_*^+$ is all of order 2. In $\mathcal{O}_*^-$ every class is of order 2, for if T is orientation reversing,

$$(M \cup M, T \cup T) \cong \partial(M \times I, T \times \mathrm{Id})$$

under the diffeomorphism $\mathrm{Id} \cup T$.

$\mathcal{O}_*^+$ was also studied by Conner [**2**], and $\mathcal{O}_*$ has recently been studied by Lee and Wasserman [**4**], who show how classes of $\mathcal{O}_*$ are detected by equivariant characteristic numbers. Our results complement these two papers, and we offer a short proof (Lemma 4) of a result which appears in both.

THEOREM 1. *Let I_* be the cobordism of unoriented manifolds with involution* [**3**, *Chapter* IV]. *The forgetful map $\rho: \mathcal{O}_* \to I_*$ is monic on 2-torsion.*

From [**6**] it follows that the torsion is mapped monomorphically into the Wall bordism group $\tilde{W}_*(\bigvee_{j=1}^{\infty} TBO(j))$ (see [**7**, Chapter VIII]). An element of this group is represented by a vector bundle $\nu \to F$, whose disk bundle $D\nu$ is a Wall manifold [**8**], together with a classifying map

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$(D\nu, S\nu)\to(\bigvee TBO(j), \infty)$. Define $\partial'[D\nu]=[Di^*\nu]$, where $i:N\subset F$ includes a submanifold dual to $w_1F+w_1\nu$. Then $Di^*\nu$ is dual to $w_1D\nu$, hence is oriented.

THEOREM 2. $\mathcal{O}_*^-\cong \operatorname{Im}\partial'\cap\tilde{W}_*(\bigvee_{j=1}^{\infty} TBO(2j+1))$.

Let $p:\tilde{W}_*(\bigvee TBO(2j))\to W_*$ map $[D\nu]$ to the class of the projective space bundle $P\nu$, and $q:\tilde{W}_*(TBO(1))\to \tilde{W}_*(\bigvee TBO(2j))$ map $[D\nu]$ to $[D(\nu\oplus 1)]$. Then $pq=0$ and $\partial'(\operatorname{Im} q)\subseteq \operatorname{Im} q$, so p and ∂' induce

$$p':\tilde{W}_*(\bigvee TBO(2j))/\operatorname{Im} q\to W_*$$

and

$$\partial'':\tilde{W}_*(\bigvee TBO(2j))/\operatorname{Im} q\to \tilde{W}_*(\bigvee TBO(2j))/\operatorname{Im} q.$$

THEOREM 3. torsion $\mathcal{O}_*^+\cong \operatorname{Im}\partial''\cap\operatorname{Ker} p'$.

These theorems are used to describe Z_2-generators of $\mathcal{O}_*^-$ and torsion $\mathcal{O}_*^+$. The usefulness of the latter description is limited by lack of knowledge of $\operatorname{Ker} p'$.

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2. **Wall manifolds with involution.** A *Wall manifold with involution*, (W, T, f), is an unoriented manifold W with an involution T and a map $f:W\to S^1$ such that $fT=f$ and $f^*(\iota)=w_1W$, where $\iota\in H^1(S^1;Z_2)$ is the generator. To construct examples one uses:

LEMMA 1 (See [**6**, (2.5)]). *If $D\nu$ is a Wall manifold (respectively, orientable) and* $\dim\nu$ *is even, then $P\nu$ is a Wall manifold (respectively, orientable).*

Let W_*^I and W_*^F be the cobordism groups of Wall manifolds with involution and with fixed point free involution, respectively. Following are the chief results of [**6**]. The forgetful map $W_*^I\to I_*$ is monic. Clearly $\operatorname{Im}\rho\subseteq W_*^I$. If $d:W_*^I\to\mathcal{O}_*$ assigns to $[W, T, f]$ the class of $(N, T|N)$, where $N\subset W$ is an invariant submanifold dual to w_1W, then the sequence

$$\mathcal{O}_*\xrightarrow{\rho} W_*^I\xrightarrow{d}\mathcal{O}_*\xrightarrow{2}\mathcal{O}_*$$

is exact. Define $d'=\rho d:W_*^I\to W_*^I$. The map $s:W_*^I\to\tilde{W}_*(\bigvee TBO(j))$, given by classifying the normal bundle to the fixed set, is monic. The map $\partial:\tilde{W}_*(\bigvee TBO(j))\to W_*^F$, given by $\partial[D\nu]=[S\nu$, antipodal map], is epic.

We give a new proof of the last fact by constructing a splitting map. Recalling [**6**, (3.1)], if $[M,f]\in W_*(RP^\infty)$ let $\pi:\tilde{M}\to M$ be the double cover classified by f. If $[M,f]\in N_*(S^1)$ let $\pi:\tilde{M}\to M$ be the double cover corresponding to $f^*\iota+w_1M$. In either case $\tilde{M}$ is a Wall manifold with

involution Δ, the deck transformation reversing sheets of $\tilde{M}$. This defines monomorphisms

$$k: W_*(RP^\infty) \to W_*^F, \qquad j: N_*(S^1) \to W_*^F$$

with $k+j$ epic and $\operatorname{Im} k \cap \operatorname{Im} j \cong W_*$, identified as the set of classes $[W \times S^0, \operatorname{Id} \times \Delta]$ for $[W] \in W_*$.

Now let $\lambda \to M$ be the line bundle corresponding to π. If $[M, f] \in N_*(S^1)$, $D\lambda$ is a Wall manifold. Define

$$\delta[\tilde{M}, \Delta] = [D\lambda] \in \tilde{W}_*(\vee\, TBO(n)).$$

If $[M, f] \in W_*(RP^\infty)$, let $i: N \subset M$ include a submanifold defining the double cover. $D(i^*\lambda)$ is a Wall manifold; by Lemma 1 so is $P = P(i^*\lambda \oplus 1)$. Define

$$\delta[\tilde{M}, \Delta] = [D(i^*\lambda \oplus 1)] + [D(1 \to M)].$$

LEMMA 2. $\partial\delta = \operatorname{Id}: W_*^F \to W_*^F$.

PROOF. For $x \in \operatorname{Im} j$ it is obvious that $\partial\delta(x) = x$. If $[M, f] \in W_*(RP^\infty)$ write $P = D(i^*\lambda) \cup D(i^*\lambda)$. The map $g: P \to RP^\infty$, classifying the canonical line bundle, may be written

$$g = (f|N)p_0 \cup *\tag{1}$$

where p_0 is the projection of $D(i^*\lambda)$ and $*$ is homotopic to a trivial map (see the proof of [3, (26.1)]). Since $i^*\lambda$ is the normal bundle of N in M we may also assume f is of the form (1). Now form (H, h) from $(M \times I, f\pi_M)$ and $(P \times I, g\pi_P)$ by sewing copies of $(D(i^*\lambda) \times 1, (f|N)p_0)$.

$\partial(H, h) = (M, f) \cup (P, g) \cup (M, *)$, which proves $\partial\delta[\tilde{M}, \Delta] = [\tilde{M}, \Delta]$. □

3. **Proof of Theorem 1.** The statement of the theorem can be sharpened:

THEOREM 1′. *ρ maps* torsion $\mathcal{O}_*$ *isomorphically onto* $\operatorname{Im} d'$.

LEMMA 3. *Let (M, T) be an orientable manifold with involution, and $(M, T) = \partial(Q, U)$ as unoriented manifolds with involution. If Q is in fact orientable, then*

$$[M, T] = 2\beta + \omega[S^0, \Delta]$$

for some $\beta \in \mathcal{O}_*^+$ and $\omega \in \Omega_*$.

PROOF. Give Q an orientation and let $\partial Q = M'$ be its oriented boundary.

$$[M, T] = [M, T] - [M', T].\tag{2}$$

Of course $M' = M$ up to the sign of the orientation on each component. $2x = 0$ for $x \in \mathcal{O}_*^-$; also the involution $M \cup (-M) \to M \cup (-M)$, reversing components, bounds $(M \times [-1, 1], \operatorname{Id} \times (-1))$.

(2) then reduces to the desired form. □

LEMMA 4. *If $x \in \mathcal{O}_n$ and $\rho x=0$ then either $x \in 2\mathcal{O}_n$ or else $n=4i$ and*

$$x = 2\beta + \omega[S^0, \Delta] \tag{3}$$

for some $\beta \in \mathcal{O}^+_{4i}$ and $\omega \in \Omega_{4i}$/torsion.

Note. This is [**4**, Proposition 21] and, essentially, [**2**, Theorem 5.8]. Following is a short proof relying on [**6**].

PROOF. Write $x=y+z$, $y \in \mathcal{O}^+_*$, $z \in \mathcal{O}^-_*$. Since $s\rho y$ and $s\rho z$ lie in orthogonal summands we must have $\rho y=0=\rho z$.

Let (M, T) represent either y or z, and let (W, S, f) be a Wall manifold with involution and boundary $(M, T, *)$. By [**6**, (4.1)] we may assume $f(M)=\{a\}$ and f is transverse regular at $b \neq a$. If $b \in \operatorname{Im} f$, cut W along $N=f^{-1}b$, as in [**8**, §1], obtaining an oriented cobordism of (M, T) with $(N \times S^0, (S|N) \times \mathrm{Id})$. This bounds if S is orientation reversing. If $b \notin \operatorname{Im} f$, then W is orientable and Lemma 3 applies. Finally, Rosenzweig showed [**5**] that $2\beta=0$ if $n \neq 4i$ and that $\omega[S^0, \Delta]=0$ if $2\omega=0$. □

Suppose $x \in \mathcal{O}_n$, $\rho x=0=2x$. If $x \in 2\mathcal{O}_n$ then $x=0$, for the only torsion is of order 2. If x has the form (3), then since $2x=0$ we have $4\beta=2\omega[S^0, \Delta]$. By the proof of [**2**, Corollary 5.5], index (ω) is even. Then [**2**, Theorem 4.5] proves $\omega[S^0, \Delta] \in 2\mathcal{O}^+_{4i}$. Thus $x=0$ in this case also, and ρ is monic on the torsion. The reader can easily show that $\rho(\text{torsion})=\operatorname{Im} d'$, completing Theorem 1′. □

4. **Calculation of ∂'.** To give meaning to Theorems 2 and 3, we compute the homomorphism ∂'. The structure of $\tilde{W}_*(\bigvee TBO(n))$ is determined by [**3**, §17] and [**7**, pp. 163–164].

LEMMA 5. *$\tilde{W}_*(\bigvee TBO(n))$ is the W_*-polynomial algebra on generators $[D\lambda_i]$, one of each dimension $i \geqq 1$, with λ_i a line bundle. If $i=2m+1$, we may take λ_i to be the canonical line bundle over RP^{2m}. If $i=2m+2$, we may take λ_i to be the canonical line bundle over $P(\xi \oplus 2m \to S^1)$, where $\xi \to S^1$ is the nontrivial line bundle.*

The proof is perfectly straightforward, and all the needed cohomology data is listed in [**6**, (4.1)].

Recall that $\partial'[D\nu]=[Di^*\nu]$, if $\nu \to F$ and $i: N \subset F$ includes a submanifold dual to $w_1\nu+w_1F$. By [**7**, pp. 169–172], $D\nu$ is orientable if and only if $[D\nu] \in \operatorname{Im} \partial'$. For future reference, if ν has even fiber dimension then by [**6**, (2.4)] it is also true that $P(i^*\nu)$ is dual in $P\nu$ to $w_1P\nu$. Then

$$2[P(i^*\nu)] = 0 \in \Omega_*,$$

by [**8**].

LEMMA 6. *For all* m, $\partial'[D\lambda_{2m+2}]=[D\lambda_{2m+1}]$ *and* $\partial'[D\lambda_{2m+1}]=0$. *If* $\partial_1: W_* \to W_*$ *maps* W *to the submanifold dual to* $w_1 W$ [**8**], *then*

$$\partial'(w\beta) = (\partial_1 w)\beta + w(\partial'\beta)$$

for all $w \in W_*$, $\beta \in \tilde{W}_*(\bigvee TBO(n))$.

REMARKS. The first statement is immediate and the second may be proved with characteristic numbers, exactly like [**8**, Lemma 3]. From the proof of [**6**, (4.1)], we see that $\partial' s = s d'$. Since s is a W_*-monomorphism, the lemma also proves that $d'(w\beta)=(\partial_1 w)\beta + w(d'\beta)$ for $w \in W_*$, $\beta \in W_*^I$.

5. **Proof of Theorems 2 and 3.** If ν has odd fiber dimension, then by [**6**, (3.3)] we have $\delta\partial[D\nu]=[D\lambda]$, where $\lambda \to P\nu$ is the canonical line bundle. By Lemma 1, $P(\nu \oplus 1)$ is a Wall manifold. Let $t(\nu)$ be the involution on $P(\nu\oplus 1)$ induced by the linear map $1\times(-1)$ in each fiber of $\nu\oplus 1$, and define

$$\sigma[D\nu] = [P(\nu \oplus 1), t(\nu)] \in W_*^I.$$

Then $s\sigma[D\nu]+\delta\partial[D\nu]=[D\nu]$, just as in [**3**, §28]. It follows that $\sigma[D\nu]=0$ if and only if $[D\nu]=\delta\partial[D\nu] \in \tilde{W}_*(TBO(1))$. Also, since $\partial s=0$, it follows that $\sigma s\rho(x)=\rho(x)$ for $x \in \mathcal{O}_*^-$. Since $P(\nu\oplus 1)$ is orientable if $D\nu$ is, and $t(\nu)$ is then orientation reversing, Theorem 2 is proved.

If $\nu \to F$ has even fiber dimension then we will assume $[D\nu] \in \operatorname{Ker} p$. To justify this restriction, suppose $[M, T] \in \mathcal{O}_*^+$ and ν is the normal bundle to the fixed set of T. Then $P\nu$ bounds $(M - \operatorname{Int} D\nu)/T$, so $ps\rho[M, T]=0$.

Suppose, then, that $[D\nu] \in \operatorname{Ker} p$. Since $P\nu$ is a Wall manifold, Lemma 2 implies $\delta[D\nu]=[D(i^*\lambda\oplus 1)]$, where $i: N \subset P\nu$ includes a submanifold dual to λ. We define $\sigma: \operatorname{Ker} p \to W_*^I$. Let $\tilde{N}=S(i^*\lambda)$, so $\tilde{N}\to N$ is a double cover and $\tilde{N}$ is imbedded in $S\nu$ with trivial normal bundle. $\tilde{N}$ is also imbedded in $S(i^*\lambda\oplus 1)$ with trivial normal bundle, so we sew $D\nu$ and $D(i^*\lambda\oplus 1)$ along the copies of $\tilde{N}\times I$ in the boundaries. The remaining boundary is then two copies of $P\nu$, which bounds some Q. Sewing on $2Q$ builds a Wall manifold $K(\nu)$, which admits an involution $t(\nu)$ that restricts to the antipodal involution on the disk bundles, and reverses the two copies of Q. Thus the fixed set of $t(\nu)$ is the disjoint union of F and N, and the normal bundles are ν and $i^*\lambda\oplus 1$, respectively.

Define $\sigma[D\nu]=[K(\nu), t(\nu)] \in W_*^I$; it is easy to see that this is a well-defined homomorphism. By construction,

$$s\sigma[D\nu] + \delta\partial[D\sigma] = [D\nu].$$

Reasoning as before, $\sigma[D\nu]=0$ if and only if $[D\nu]=\delta\partial[D\nu] \in \operatorname{Im} q$, and $\sigma s\rho(x)=\rho(x)$ for $x \in \mathcal{O}_*^+$.

Finally, suppose $D\nu$ is oriented. Then so are $D(i^*\lambda\oplus 1)$ and $P\nu$. As noted in the last section, $P\nu$ represents a torsion class in Ω_*. Since $\Omega_*\to W_*$ is monic on torsion [**8**] we may assume Q is oriented. Thus $K(\nu)$ is oriented, and $t(\nu)$ is orientation preserving. This completes the proof of Theorem 3.

REMARKS. (1) We have shown that $\mathcal{O}_*^-$ is the Z_2-vector space on generators $[P(\nu\oplus 1), t(\nu)]$, for $[D\nu]\in \operatorname{Im}\partial'$, $\dim \nu=2m+1$, $m>0$. The torsion of $\mathcal{O}_*^+$ is a Z_2-vector space on generators $[K(\nu), t(\nu)]$ for

$$[D\nu]\in \operatorname{Im}\partial''\cap \operatorname{Ker} p',$$

but this is not very helpful, since we cannot actually construct $K(\nu)$ without knowing $Q=\partial P\nu$.

(2) If $[M, T]\in \mathcal{O}_*^-$, then clearly $[M]$ is a torsion class in Ω_*. Conversely, P. G. Anderson proved [**1**, Proposition 5] that every torsion class in Ω_* is represented by one of our examples $P(\nu\oplus 1)$.

REFERENCES

1. P. G. Anderson, *Cobordism classes of squares of orientable manifolds*, Ann. of Math. (2) **83** (1966), 47–53. MR **32** #4700.

2. P. E. Conner, *Orientation-preserving involutions*, Lectures on the action of a finite group, Lecture Notes in Math., no. 73, Springer-Verlag, Berlin, 1968, Chap. 2. MR **41** #2670.

3. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Heft 33, Academic Press, New York; Springer-Verlag, Berlin, 1964, MR **31** #750.

4. C. N. Lee and A. G. Wasserman, *Equivariant characteristic numbers*, Proc. Second Conf. Transformation Groups, part I, Lecture Notes in Math., vol. 298, Springer-Verlag, Berlin, 1972, pp. 191–216.

5. H. L. Rosenzweig, *Bordism of involutions on manifolds*, Illinois J. Math **16** (1972)., 1–10.

6. R. J. Rowlett, *Wall manifolds with involution*, Trans. Amer. Math. Soc. **169** (1972), 153–162.

7. R. E. Stong, *Notes on cobordism theory*, Princeton Univ. Press, Princeton, N.J.; Univ. of Tokyo Press, Tokyo, 1968. MR **40** #2108.

8. C. T. C. Wall, *Determination of the cobordism ring*, Ann. of Math. (2) **72** (1960), 292–311. MR **22** #11403.

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