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ON HYPERFINITE W-* ALGEBRAS

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ABSTRACT. If $\mathscr{A}$ is a W-* algebra on separable Hilbert space H, and if $\mathscr{A}(\lambda)$ are the factors in the direct integral decomposition of $\mathscr{A}$, then $\mathscr{J}=\{\lambda|\mathscr{A}(\lambda)$ is hyperfinite$\}$ is μ-measurable, and $\mathscr{A}$ is hyperfinite if and only if $\mathscr{A}(\lambda)$ is hyperfinite μ-a.e.

Let $\mathscr{A}$ be a W-* algebra on separable Hilbert space H. $\mathscr{A}$ is *hyperfinite* if there is an increasing sequence of finite dimensional W-* subalgebras $\mathscr{A}_n$ of $\mathscr{A}$ whose union generates $\mathscr{A}$. Let $\mathscr{A}=\int_\Lambda \oplus \mathscr{A}(\lambda)\mu(d\lambda)$ denote the direct integral decomposition of $\mathscr{A}$ into factors, and let $\mathscr{J}=\{\lambda|\mathscr{A}(\lambda)$ is hyperfinite$\}$. We prove in this paper that $\mathscr{J}$ is μ-measurable, and that $\mathscr{A}$ is hyperfinite if and only if $\mu(\Lambda-\mathscr{J})=0$.

In dealing with direct integrals, we use the following notation (see [2], [3] for details). K will denote the underlying separable Hilbert space of H. Letting d denote a metric on $\mathscr{B}(K)$ which induces the strong operator topology on bounded subsets of $\mathscr{B}(K)$ [2, Lemma I.4.9], define $M(T)=d(T,0)$. Then a bounded sequence $T_n\in\mathscr{B}(K)$ converges strongly to 0 if and only if $M(T_n)\to 0$.

By $\mathscr{S}$ we denote the unit ball of $\mathscr{B}(K)$ taken with the strong -* topology. $\mathscr{S}_n$, n an integer, denotes the n-fold Cartesian product of $\mathscr{S}$. Finally, let $B_n\in\mathscr{A}$ be a sequence in the unit ball of $\mathscr{A}$ such that $\{B_n(\lambda)\}$ is strong-* dense in the unit ball of $\mathscr{A}(\lambda)$ μ-a.e., and such that $B_n(\lambda)$ is strong-* continuous in λ.

Before proving our main results, we consider the structure of a finite dimensional W-* algebra $\mathscr{B}$. Since any finite dimensional linear space of operators is strongly closed, $\mathscr{B}$ is finite dimensional if and only if there is a finite set of operators $T_1,\cdots,T_n$ such that each product T_iT_j and each adjoint T_k^* is a linear combination of the T_M.

By the Kaplansky Density Theorem, it suffices for these operators to be strong limits of such linear combinations having bounded norms and coefficients in C_0, the set of complex numbers with rational real and imaginary parts. We apply this idea to define hyperfiniteness through countably many conditions. Indeed, if $\mathscr{A}$ is hyperfinite, for each n there

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are n operators whose linear span is a finite dimensional W-* algebra $\mathscr{A}_n$ such that the $\mathscr{A}_n$ form an increasing sequence (although not necessarily strictly increasing) whose union generates $\mathscr{A}$. This explains the conditions in Theorem 1.

THEOREM 1. *Let* $\mathscr{J}=\{\lambda | A(\lambda)$ *is hyperfinite*$\}$. *Then* $\mathscr{J}$ *is* μ-*measurable.*

PROOF. Let $\mathscr{P}=\Lambda\times\prod_{n=1}^{\infty}\mathscr{S}_n$, let π denote the projection of $\mathscr{P}$ onto Λ, and let $T(n)=(T(n, 1), \cdots, T(n, n))$ denote a typical element of $\mathscr{S}_n$. Consider the following conditions on elements $[\lambda, T(n)]$ of $\mathscr{P}$:

(1) $T(n, m)\in\mathscr{A}(\lambda)$.

(2) For some $T=\sum_{k=1}^{n} a_k T(n, k), a_k\in \boldsymbol{C}_0, T\in\mathscr{S}$, and $M(T(n,i)T(n,j)-T) <1/r$.

(3) For some $T=\sum_{k=1}^{n} b_k T(n, k)$, $b_k\in \boldsymbol{C}_0$, $T\in\mathscr{S}$, and $M(T(n, i)^*-T)< 1/r$.

(4) For some $T=\sum_{k=1}^{n+1} c_k T(n+1, k)$, $c_k\in \boldsymbol{C}_0$, $T\in\mathscr{S}$, and $M(T(n, i)-T) <1/r$.

(5) For some $T=\sum_{k=1}^{p} d_k T(p, k)$, $d_k\in \boldsymbol{C}_0$, $T\in\mathscr{S}$, and $M(B_n(\lambda)-T)< 1/r$.

It is easy to see that if $\mathscr{J}'$ is the subset of $\mathscr{P}$ for which condition (1) holds for every m and n and the remaining conditions hold for every r, etc. for appropriate coefficients, then $\mathscr{J}'$ is μ-measurable and $\pi(\mathscr{J}')$ differs from $\mathscr{J}$ by a μ-null set. Hence, by [**2**, Lemma I.4.6], $\mathscr{J}$ is μ-measurable. Q.E.D.

THEOREM 2. $\mathscr{A}$ *is hyperfinite if and only if* $\mu(\Lambda-\mathscr{J})=0$.

PROOF. Suppose $\mathscr{A}$ is hyperfinite. Then for each n there is a finite dimensional W-* subalgebra $\mathscr{A}_n$ of $\mathscr{A}$ which is the linear span of $T(n, 1), \cdots, T(n, n)\in\mathscr{A}$; these algebras form an increasing sequence whose union generates $\mathscr{A}$. Now let $\mathscr{A}_n(\lambda)$ be the W-* algebra generated by $T(n, i)(\lambda)$, $i=1, \cdots, n$. By [**5**, Lemma 1] it follows that $\mathscr{A}_n(\lambda)$ is an increasing sequence of finite dimensional W-* algebras contained in $\mathscr{A}(\lambda)$ for μ-a.e. λ. Since $B_n\in\mathscr{A}$ and the $\mathscr{A}_n$ generate $\mathscr{A}$, a second application of [**5**, Lemma 1] shows that the $\mathscr{A}_n(\lambda)$ generate $\mathscr{A}(\lambda)$ μ-a.e. Thus $\mu(\Lambda-\mathscr{J})=0$.

Conversely, suppose that $\mu(\Lambda-\mathscr{J})=0$. Using the proof of Theorem 1 and [**2**, Lemma I.4.7] we can construct an increasing sequence of finite dimensional W-* subalgebras $\mathscr{A}_n$ of $\mathscr{A}$ such that $\mathscr{A}_n(\lambda)$ generate $\mathscr{A}(\lambda)$ μ-a.e. It follows that $\mathscr{A}$ is generated by the $\mathscr{A}_n$ and $\mathscr{Z}$, the center of $\mathscr{A}$. But $\mathscr{Z}$ is hyperfinite [**4**, Lemma 2], and if $\mathscr{C}_n$ is an increasing sequence of finite dimensional W-* algebras generating $\mathscr{Z}$, then clearly for each n the algebra $\mathscr{D}_n$ generated by $\mathscr{A}_n$ and $\mathscr{C}_n$ is finite dimensional. Since $\mathscr{A}$ is generated by the increasing sequence $\mathscr{D}_n$, the result is proved. Q.E.D.

We remark in conclusion that the idea of hyperfinite algebras was introduced by Murray and von Neumann in **[1]** to treat factors of type II_1. They proved that in this case the $\mathscr{A}_n$ could be chosen to be factors of type I_{2^n}. This result has recently been extended to hyperfinite factors of types II_∞ and III by E. J. Woods and G. Elliott (private communication). It is easy to see that our methods could then show that, modulo the center $\mathscr{Z}$, $\mathscr{A}$ hyperfinite is generated by a sequence of factors of type I_{2^n}.

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