PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 1, September 1973

# A CLASS OF $C^*$-ALGEBRAS

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ABSTRACT. In this paper we present a construction of $C^*$-algebras based on a countable system of subsets of a countable set. The properties of such algebras are described in terms of this system of sets.

In [1] Behncke, Krauss and Leptin suggested a generalization of their construction of $C^*$-algebras. In this note this program is carried through and all results of [1] are extended to this more general case.

This note is organized as follows: In the first part we construct for a given system $M$ of subsets of a countable set $m_0$ a $C^*$-algebra $\mathfrak{A}(M)$. Then the properties of $\mathfrak{A}(M)$ for a finite $m_0$ are studied. These results are the key to the remainder. With their aid the general case can be investigated. We conclude this note with two examples.

Throughout all sets will be countable and all Hilbert spaces will be complex separable spaces. If $H$ is a Hilbert space, the algebra of all compact (bounded) linear operators will be denoted by $\mathfrak{K}(H)$ ($\mathfrak{B}(H)$). The term ideal always means closed two sided ideal. If $m$ is a set, the complement of $m$ will be denoted by $m'$.

I. Let $m_0$ be a countable set and let $M$ be a countable system of subsets of $m_0$. For any $m \subset m_0$ let $\tilde{m}$ denote the set of all functions on $m$ with values in $\{0, 1, 2, \ldots\}$ and finite support. Then every decomposition of $m_0$ into subsets $m$ and $m'$, $m_0 = m \cup m'$, induces a decomposition of $\tilde{m}_0$, $\tilde{m}_0 = \tilde{m} \times \tilde{m}'$. Thus with $H_m = l^2(\tilde{m})$ we get

$$H = H_{m_0} = H_m \otimes H_{m'} \quad \text{for all } m \subset m_0.$$

With this define $\mathfrak{K}_m = 1_{H_m} \otimes \mathfrak{K}(H_{m'})$ and $\mathfrak{K}_{m_0} = (0)$. Then $\mathfrak{A}(M)$ will denote the $C^*$-algebra of operators on $H$, which is generated by all $\mathfrak{K}_m$ with $m \in M$. If $N \subset M$ then $\mathfrak{A}(N)$ denotes the subalgebra of $\mathfrak{A}(M)$ which is generated by all $\mathfrak{K}_n$ with $n \in N$. Throughout this paper $M$ will be fixed. Since $M$ is countable, $\mathfrak{A}(M)$ is separable. The algebras $\mathfrak{K}_m$ satisfy the relations

$$\overline{\mathfrak{K}_m \cdot \mathfrak{K}_n} = \overline{\mathfrak{K}_n \cdot \mathfrak{K}_m} = \mathfrak{K}_{m \cap n}. \tag{1}$$

Received by the editors October 23, 1972.
*AMS (MOS) subject classifications* (1970). Primary 46L05.
*Key words and phrases.* $C^*$-algebras, operator algebras.

If $n_0=\bigcap_{m\in M} m\neq\varnothing$, we can split off from every $x\in\mathfrak{A}(M)$ the common tensorfactor $1_{H_{n_0}}$. Thus without loss of generality we may restrict $M$ by the conditions

$$(2)\qquad m, n\in M\Rightarrow m\cap n\in M \quad\text{and}\quad \bigcap_{m\in M} m=\varnothing.$$

Then we can show as in [**1**].

THEOREM 1. *$\mathfrak{A}(M)$ acts irreducibly on $H$.*

II. Throughout this section we assume that $m_0$ is a finite set.

LEMMA 1. *Let $x=\sum_{m\in M} x_m$ with $x_m\in\mathfrak{K}_m$ and let $n$ be a maximal element of $\{m\in M|x_m\neq 0\}$. Then $|x|\geqq|x_n|$.*

PROOF. (a) Since $|xx^*|=|x|^2$ and since $n$ is maximal, we may assume $x\geqq 0$ and $x_n\geqq 0$. Let $y\in\mathfrak{K}_n$ be the spectral projection of $x_n$ for the eigenvalue $|x_n|$. Then $|x|\geqq|yxy|$ and $|x_n|\,y=yx_ny$. Thus it suffices to show $|yxy|\geqq|x_n|$, and we may assume even $yxy=x$ and $x_n=y$. Since $yx_my\in\mathfrak{K}_m\cap\mathfrak{K}_n$ we have $x_m\neq 0$ only if $m\subset n$. Considering the construction of $\mathfrak{A}(M)$ we may therefore write $m_0=\{1,\cdots,j\}$ and $n=\{1,\cdots,j-1\}$.

(b) Thus we have $H=\otimes_{i=1}^{j} H_i$ and $H_m=\otimes_{i\in m} H_i$ with $H_i=l^2(N)$. Since $\mathfrak{K}_m=1_{H_m}\otimes(\otimes_{i\in m'}\mathfrak{K}(H_i))$ we can find for each $x_m$ with $m\subset n$ and $\varepsilon>0$ an element $x'_m\in\mathfrak{A}(M)$ with the properties:

(i) $x'_m$ is a finite sum of elements of the form $1_{H_m}\otimes(\otimes_{i\in m'} x_i)$ where

(ii) $yx'_my=x'_m$,

(iii) each $x_i\in\mathfrak{K}(H_i)$ has finite rank,

(iv) $|x_m-x'_m|<\varepsilon 2^{-j}$.

Thus for $i\in m'$ there exists a projection $p_{i,m}\in\mathfrak{K}(H_i)$ such that $(1_{H_{\{i\}'}}\otimes p_{i,m})x'_m=x'_m$. For $i\in m_0$ let now $p_i=\bigvee p_{i,m}\in\mathfrak{K}(H_i)$ where the join is taken over all $m$ with $i\in m$. Because of (ii) we can even achieve $1_{H_n}\otimes p_j=y$. For $m\in M$ let now $p_m=1_{H_m}\otimes(\otimes_{i\in m'} p_i)\in\mathfrak{K}_m$. Then all $p_m$ commute and the projection $p=\prod_{m\neq n}(y-p_m)\in\mathfrak{A}(m)$ satisfies $px'_mp=0$ for all $m$ with $m\subsetneq n$ and $x_np=p\neq 0$.

(c) Thus we have $|x|\geqq|pxp|\geqq|px_np|-\sum_{m\neq n}|p(x_m-x'_m)p|>|x_n|-\varepsilon$. Since $\varepsilon>0$ is arbitrary we have $|x|\geqq|x_n|$.

With the same arguments as in [**1**] we can now show:

COROLLARY. *Every $x\in\mathfrak{A}(M)$ has a unique expansion $x=\sum_{m\in M} x_m$ with $x_m\in\mathfrak{K}_m$.*

Now let $\mathfrak{J}$ be an ideal in $\mathfrak{A}(M)$. If for some $n\in M$ we have $\mathfrak{K}_n\cap\mathfrak{J}\neq(0)$, then $\mathfrak{K}_n\subset\mathfrak{J}$. Hence we may define the set $M(\mathfrak{J})=\{m\in M|\mathfrak{K}_m\subset\mathfrak{J}\}$. Due to (1), $M(\mathfrak{J})$ satisfies

$$(3)\qquad m\in M(\mathfrak{J}),\ n\in M\Rightarrow m\cap n\in M(\mathfrak{J}).$$

Subsets of $M$ with (3) we call therefore ideal systems. Conversely if $N \subset M$ is an ideal system, the algebra $\mathfrak{A}(N)$ is an ideal of $\mathfrak{A}(M)$. This follows easily from the corollary. Now let $\mathfrak{J}$ be a primitive ideal and assume $M(\mathfrak{J})'$ contains two distinct minimal elements $m_1$ and $m_2$. Then the algebras $\mathfrak{J}_i = \mathfrak{J} + \mathfrak{K}_{m_i}$ with $i=1, 2$ are ideals with $\mathfrak{J}_1 \cdot \mathfrak{J}_2 = \mathfrak{J}$. This however is impossible [**2**, 2.11.4]. Hence in this case $M(\mathfrak{J})$ satisfies in addition to (3):

$$m, n \in M(\mathfrak{J})' \Rightarrow m \cap n \in M(\mathfrak{J})'. \tag{4}$$

Here $M(\mathfrak{J})'$ denotes the complement of $M(\mathfrak{J})$. Ideal systems with (4) we call thus $p$-ideal systems.

LEMMA 2. (i) *There is a one to one correspondence between p-ideal systems and primitive ideals. If $\mathfrak{J}$ is a primitive ideal, $M(\mathfrak{J})$ is a p-ideal system and $\mathfrak{J} = \mathfrak{A}(M(\mathfrak{J}))$.*

(ii) *Let $N \subset M$ be a p-ideal system and let $n_0$ be the minimal element in $N'$, then $\mathfrak{A}(N)$ is a primitive ideal of $\mathfrak{A}(M)$ and we have $N = M(\mathfrak{A}(N))$. Furthermore*

$$\mathfrak{A}(M) = \mathfrak{A}(N) \oplus \mathfrak{A}(N') \quad \textit{and} \quad \mathfrak{A}(M)/\mathfrak{A}(N) \cong \mathfrak{A}(N') \cong \mathfrak{A}(N_0'), \tag{5}$$

*where $N_0' = \{n \cap n_0' | n \in N'\}$ is considered as a system of subsets of $n_0'$.*

*Moreover we have, for any $x = \sum_{n \in M} x_n$ with $x_n \in \mathfrak{K}_n$,*

$$|x| \geqq \left| \sum_{n \in N'} x_n \right|. \tag{6}$$

PROOF. (i) Clearly $\mathfrak{A}(M(\mathfrak{J})) \subset \mathfrak{J}$ and if inequality holds, there exists an element $0 \neq x = \sum_{n \in M(\mathfrak{J})'} x_n \subset \mathfrak{J}$. Since $\mathfrak{J}$ is primitive, $M(\mathfrak{J})'$ has a single minimal element $n_0$ due to (4). Let $y \in \mathfrak{K}_{n_0}$, then $xy \in \mathfrak{K}_{n_0} \cap \mathfrak{J}$. If $xy \neq 0$ then $\mathfrak{K}_{n_0} \subset \mathfrak{J}$ or $n_0 \in M(\mathfrak{J})$. This however is impossible, because $n_0 \in M(\mathfrak{J})'$. Hence we have $\mathfrak{K}_{n_0} \cdot x = 0$. Let $N = M(\mathfrak{J})$; then each $z \in \mathfrak{A}(N')$ has the form $1_{H_{n_0}} \otimes \tilde{z} = z$. The isomorphism $z \to \tilde{z}$ amounts to a representation of $\mathfrak{A}(N')$ as $\mathfrak{A}(N_0')$, with $N_0'$ as above. The image of $\mathfrak{K}_{n_0}$ under this isomorphism is the algebra of all compact operators. Hence $\mathfrak{K}_{n_0} \cdot x = 0$ implies $x = 0$.

(ii) The corollary above shows $\mathfrak{A}(M) = \mathfrak{A}(N) \oplus \mathfrak{A}(N')$. The canonical homomorphism $\pi: \mathfrak{A}(M) \to \mathfrak{A}(M)/\mathfrak{A}(N)$ maps $\mathfrak{A}(N')$ faithfully, because $\pi$ is faithful on $\mathfrak{K}_{n_0}$. Hence $\mathfrak{A}(M)/\mathfrak{A}(N) \cong \mathfrak{A}(N')$. The isomorphism $\mathfrak{A}(N') \cong \mathfrak{A}(N_0')$ follows with the arguments of (i). This shows also $|x| \geqq |\pi(x)| = |\sum_{n \in N'} x_n|$ or (6).

COROLLARY. *There is a one to one correspondence between ideals of $\mathfrak{A}(M)$ and ideal systems. For any ideal $\mathfrak{J}$ we have $\mathfrak{J} = \mathfrak{A}(M(\mathfrak{J}))$ and for any ideal system $N$ we have $N = M(\mathfrak{A}(N))$.*

PROOF. The ideal $\mathfrak{J}$ is the intersection of the primitive ideals $\mathfrak{J}_i$, which contain it, $\mathfrak{J}=\bigcap_{i=1}^k \mathfrak{J}_i$ with $k<\infty$. Then $M(\mathfrak{J})=\bigcap_{i=1}^k M(\mathfrak{J}_i)$ and $\mathfrak{A}(M(\mathfrak{J}))=\bigcap_{i=1}^k \mathfrak{A}(M(\mathfrak{J}_i))=\bigcap_{i=1}^k \mathfrak{J}_i=\mathfrak{J}$. The other statement is shown similarly.

The algebra $\mathfrak{A}(M)$ contains primitive ideals, which can be described easily in terms of $M$.

DEFINITION. For each $n \in M$, let $N_n=\{m \in M | n \not\subset m\}$ and let $\mathfrak{J}_n=\mathfrak{A}(N_n)$. Since $N_n$ is a $p$-ideal system, $\mathfrak{J}_n$ is a primitive ideal. One shows easily that the map $\psi: M \to \operatorname{Prim} \mathfrak{A}(M)$ defined by $\psi(n)=\mathfrak{J}_n$ is order preserving. Due to (5) of Lemma 2 each primitive ideal of $\mathfrak{A}(M)$ has such a form. Hence $M$ and Prim $\mathfrak{A}(M)$ are order isomorphic. More generally it is an easy consequence of our construction that order isomorphic systems $M_1$ and $M_2$ with (2) lead to isomorphic algebras $\mathfrak{A}(M_1)$ and $\mathfrak{A}(M_2)$. Hence the properties of $\mathfrak{A}(M)$ depend only on the partial order properties of $M$. This holds even if $m_0$ is infinite and if $M$ is finite.

III. Now let $m_0$ be a countably infinite set and let $M$ be a countable system of subsets of $m_0$. Keeping the notation of (II) we can show now

LEMMA 3. *Let $N \subset M$ be a p-ideal system. Then*

$$\mathfrak{A}(M) = \mathfrak{A}(N) \oplus \mathfrak{A}(N') \tag{7}$$

*and every $x \in \mathfrak{A}(M)$ has a unique decomposition $x=x_1+x_2$ with $x_1 \in \mathfrak{A}(N)$, $x_2 \in \mathfrak{A}(N')$ and*

$$|x| \geqq |x_2|. \tag{8}$$

PROOF. Let $x \in \mathfrak{A}(M)$ be an element with a finite expansion $x=\sum x_n$ with $x_n \in \mathfrak{K}_n$. Then $x$ has an obvious decomposition $x=x_1+x_2$ with $x_1 \in \mathfrak{A}(N)$ and $x_2 \in \mathfrak{A}(N')$. Then $|x| \geqq |x_2|$ is an easy consequence of (6). Since the set of elements with such a finite expansion is dense in $\mathfrak{A}(M)$ the general result follows easily as in [1] with some Cauchy-sequence argument.

THEOREM 2. (i) *Let $\mathfrak{J}$ be a primitive ideal. Then $M(\mathfrak{J})$ is a p-ideal system and $\mathfrak{A}(M(\mathfrak{J}))=\mathfrak{J}$.*

(ii) $\mathfrak{A}(M)/\mathfrak{J} \cong \mathfrak{A}(M(\mathfrak{J})') \cong \mathfrak{A}(M(\mathfrak{J})'_0)$ *where* $M(\mathfrak{J})'_0=\{m \cap n'_0 | m \in M(\mathfrak{J})'\}$ *is considered as a system of subsets of $n'_0$ with $n_0=\bigcap_{n \in M(\mathfrak{J})'} n$.*

(iii) *If $N \subset M$ is a p-ideal system, then $\mathfrak{A}(N)$ is a primitive ideal and $N=M(\mathfrak{A}(N))$.*

PROOF. (i) If $M(\mathfrak{J})$ does not satisfy (4) then there exist $n_1$ and $n_2$ in $M(\mathfrak{J})'$ with $n_1 \cap n_2 \in M(\mathfrak{J})$. Then let $\mathfrak{J}_1$ be the ideal which is generated by $\mathfrak{J}$ and all $\mathfrak{K}_n$ with $n \subset n_1$. Similarly define $\mathfrak{J}_2$. Then $\mathfrak{J} \neq \mathfrak{J}_1$, $\mathfrak{J} \neq \mathfrak{J}_2$ and $\mathfrak{J}_1 \cdot \mathfrak{J}_2=\mathfrak{J}$, which is impossible. Now let $\pi$ denote the canonical homomorphism $\mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$. Then $\ker \pi \supset \mathfrak{A}(M(\mathfrak{J}))$ and $\pi$ is faithful on each $\mathfrak{K}_n$

with $n \in M(\mathfrak{I})'$. Since $M(\mathfrak{I})$ is a $p$-ideal system, $\pi$ is a faithful homomorphism on all $x \in \mathfrak{A}(M(\mathfrak{I})')$ with a finite expansion, $x = \sum_{n \in M(\mathfrak{I})'} x_n$. Hence because of the density of such elements in $\mathfrak{A}(M(\mathfrak{I})')$, $\pi$ is faithful on $\mathfrak{A}(M(\mathfrak{I})')$ or $\mathfrak{I} = \ker \pi = \mathfrak{A}(M(\mathfrak{I}))$.

(ii) This shows also $\mathfrak{A}(M)/\mathfrak{I} \cong \mathfrak{A}(M(\mathfrak{I})')$. The remainder of (ii) is shown as in the finite case.

(iii) Lemma 3 shows $\mathfrak{A}(M) = \mathfrak{A}(N) \oplus \mathfrak{A}(N')$. As in (ii) we see

$$\mathfrak{A}(M)/\mathfrak{A}(N) \cong \mathfrak{A}(N') \cong \mathfrak{A}(N_0'),$$

where $N_0' = \{m \cap n_0' | m \in N'\}$ is considered as a system of subsets of $n_0'$, with $n_0 = \bigcap_{n \in N'} n$. Due to Theorem 1, $\mathfrak{A}(N_0')$ acts irreducibly on $H_{n_0'}$. Hence $\mathfrak{A}(N)$ is primitive. The relation $N \subset M(\mathfrak{A}(N))$ is obvious and the reverse inclusion is a simple consequence of Lemma 3.

LEMMA 4. *Let $\mathfrak{I} \subset \mathfrak{A}(M)$ be an ideal and let $x \in \mathfrak{I}$ be an element with a finite expansion $x = \sum x_m$ with $x_m \in \mathfrak{K}_m$. Then $\mathfrak{K}_m \subset \mathfrak{I}$ if $x_m \neq 0$.*

PROOF. (a) Let $M_0 \subset M$ be the smallest system of subsets of $m_0$ which is closed under intersection and contains all $m$ with $x_m \neq 0$. $M_0$ is finite and $J = \mathfrak{I} \cap \mathfrak{A}(M_0)$ is an ideal of $\mathfrak{A}(M_0)$ with $x \in J$. By the corollary of Lemma 2 we have for $J$ with respect to $\mathfrak{A}(M_0)$ the equation $J = \mathfrak{A}(M(J))$. Hence $\mathfrak{K}_m \subset J$ if $x_m \neq 0$.

LEMMA 5. *Let $\mathfrak{I}$ be an ideal of $\mathfrak{A}(M)$ and let $n \in M$ then $\mathfrak{I} \not\subset \mathfrak{I}_n$ iff $\mathfrak{K}_n \subset \mathfrak{I}$.*

PROOF. (a) If $\mathfrak{K}_n \subset \mathfrak{I}$ then $\mathfrak{I} \not\subset \mathfrak{I}_n$ because $\mathfrak{K}_n \not\subset \mathfrak{I}_n$. Conversely let $\mathfrak{I} \not\subset \mathfrak{I}_n$; then there exists an $x = x' + x'' \in \mathfrak{I}$ with $x' \in \mathfrak{I}_n$ and $0 \neq x'' \in \mathfrak{A}(N_n')$ (Lemma 3). Due to (ii) of Theorem 2, there exists a $y \in \mathfrak{K}_n$ with $x''y \neq 0$. Then $xy = x'y + x''y \in \mathfrak{I}$ and $x''y \in \mathfrak{K}_n$. Hence we may assume without loss of generality $0 \neq x'' \in \mathfrak{K}_n$. Considering $xx^* \in \mathfrak{I}$ if necessary we can even achieve $x'' \geqq 0$. Let $q$ be the spectral projection of $x''$ for the eigenvalue $|x''|$. Replacing $x$ by $(1/|x''|)qxq$ if necessary we achieve $x = x' + x''$ with $x'' = q$ and $x' = qx'q$.

(b) By (i) of Theorem 2, there exists a $y \in \mathfrak{A}(N_n)$ with a finite expansion $y = \sum y_m$ with $y_m \in \mathfrak{K}_m$ such that $|x' - y| < \frac{1}{4}$ and such that $qyq = y$. Let $M_0 \subset M$ be the smallest system of subsets of $M_0$ which is closed under intersection and contains $n$ and all $m$ for which $y_m \neq 0$. As in the proof (b) of Lemma 1 determine now a projection $p \in \mathfrak{A}(M_0)$ such that $|pyp| < \frac{1}{4}$ and such that $p = q + r$ with $qrq = r \in \mathfrak{I}_n \cap \mathfrak{A}(M_0)$. Then we have $pxp = px'p + p \in \mathfrak{I}$ and $|px'p| \leqq |p(x' - y)p| + |pyp| < \frac{1}{2}$. Then

$$z = p + \sum_{n=1}^{\infty} (px'p)^n (-1)^n \in \mathfrak{A}(M) \quad \text{and} \quad z(p + px'p) = p \in \mathfrak{I}.$$

Now Lemma 4 shows $\mathfrak{K}_n \subset \mathfrak{I}$.

COROLLARY. $M(\mathfrak{I})'=\{m \in M \mid \mathfrak{I} \subset \mathfrak{I}_m\}$.

LEMMA 6. *For every ideal $\mathfrak{I}$ of $\mathfrak{A}(M)$ we have $\mathfrak{I}=\bigcap_{n\in M(\mathfrak{I})'} \mathfrak{I}_n$.*

PROOF. Due to the corollary it suffices to show $\mathfrak{I}=\bigcap_{\mathfrak{I}\subset\mathfrak{I}_n} \mathfrak{I}_n$.

Since $\mathfrak{I}$ is the intersection of all primitive ideals which contain $\mathfrak{I}$ [**2**, 2.9.7] it suffices to show the lemma for primitive ideals only. Thus let $\mathfrak{I}$ be primitive and let $J=\bigcap_{n\in M(\mathfrak{I})'} \mathfrak{I}_n$. Then the canonical homomorphism $\pi:\mathfrak{A}(M)\to\mathfrak{A}(M)/J$ is faithful on $\mathfrak{A}(M(\mathfrak{I})')$. Hence $J\subset\mathfrak{I}$ and $J=\mathfrak{I}$ because $\mathfrak{I}\subset J$ is trivial.

THEOREM 3. *There is a one to one correspondence between ideals of $\mathfrak{A}(M)$ and ideal systems. For an ideal $\mathfrak{I}$ we have $\mathfrak{I}=\mathfrak{A}(M(\mathfrak{I}))$ and for an ideal system $N$ we have $N=M(\mathfrak{A}(N))$.*

PROOF. (a) Let $\mathfrak{I}$ be a nontrivial ideal. Then $M(\mathfrak{I})\neq\varnothing$, because otherwise $\mathfrak{I}\subset\bigcap_{n\in M} \mathfrak{I}_n$. In this case let $x\in\mathfrak{I}$ and let $\varepsilon>0$ be arbitrary. Then there exists a $y\in\mathfrak{A}(M)$ with a finite expansion $y=\sum y_m$ such that $|x-y|<\varepsilon$. Let $n=\bigcap_{y_m\neq 0} m$ then computing modulo $\mathfrak{I}_n$ we get, by (8), $|(x-y)\bmod \mathfrak{I}_n|=|y|<\varepsilon$. Hence $|x|<2\varepsilon$ or $x=0$ since $\varepsilon>0$ was arbitrary.

(b) $\mathfrak{A}(M(\mathfrak{I}))\subset\mathfrak{I}$ is an ideal of $\mathfrak{A}(M)$. Since $M(\mathfrak{A}(M(\mathfrak{I})))=M(\mathfrak{I})$ we have, by Lemma 6, $\mathfrak{A}(M(\mathfrak{I}))=\bigcap_{n\in M(\mathfrak{I})'} \mathfrak{I}_n=\mathfrak{I}$.

(c) The relation $M(\mathfrak{A}(N))=N$ is trivial.

COROLLARY. *$\mathfrak{A}(M)$ has maximal (minimal) nontrivial ideals iff $M$ has maximal (minimal) elements.*

As in [**1**] we show now:

THEOREM 4. (i) *$\mathfrak{A}(M)$ is antiliminal iff $M$ has no minimal element.*

(ii) *$\mathfrak{A}(M)$ is postliminal iff for every p-ideal system $N\subset M$, the set $N'$ has a minimal element.*

As in (II) define now the map $\psi: M\to\operatorname{Prim}\mathfrak{A}(M)$ by defining $\psi(n)=\mathfrak{I}_n=\mathfrak{A}(N_n)$ with $N_n=\{m\in M \mid n\not\subset m\}$. Then $\psi$ defines an order preserving injection of $M$ into $\operatorname{Prim}\mathfrak{A}(M)$. By (ii) of Theorem 3, $\psi$ is onto iff $\mathfrak{A}(M)$ is postliminal.

IV. Let $M$ be the set of all finite subsets of $N=\{0, 1, 2, \cdots\}$ and let $N\subset M$ be a $p$-ideal system. Since there are only finitely many elements below each $n\in N'$, the set $N'$ has a unique minimal element. Hence $\mathfrak{A}(M)$ is postliminal and $\operatorname{Prim}\mathfrak{A}(M)$ is order isomorphic to $M$.

Now let $M$ be the set of all proper cofinite subsets of $N$ and let $N\subset M$ be a $p$-ideal system. Since there are only finitely many elements of $M$ above each $n\in N$, the set $N$ contains maximal elements. Let $n$ be a maximal element of $N$, then, due to (4), $n$ is a maximal element of $M$. Thus $n$ has the form

$n=N-\{j\}=n_j$. Let $n_{i_1}, n_{i_2}, \cdots$ be the set of maximal elements of $N$. Then $N=\{m \in M | i_k \notin m$ for some $k\}$ and $N'=\{m \in M | \{i_1, i_2, \cdots\} \subset m\}$. Conversely let $p$ be a subset of $N$; then $N_p=\{m \in M | p \not\subset m\}$ is a $p$-ideal system and $\mathfrak{I}_p=\mathfrak{A}(N_p)$ is a primitive ideal. Hence $\mathfrak{A}(M)$ is antiliminal and Prim $\mathfrak{A}(M)$ is order isomorphic to the power set of $N$. In particular $\mathfrak{A}(M)$ has minimal nontrivial primitive ideals, even though $M$ has no minimal elements.

## References

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