NONINVERTIBLE KNOTS OF CODIMENSION 2

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Abstract. A proof of the noninvertibility of the pretzel knot $(25, -3, 13)$ is given which applies to the knots obtained by repeatedly spinning the pretzel knot.

1. Let $JH$ denote the integral group ring of the infinite cyclic group $H$. Conjugation in $JH$ is the linear extension of $t \rightarrow t^{-1}$, and is denoted by $\alpha \rightarrow \bar{\alpha}$.

Let $A$ be an Alexander matrix of a classical knot $k$; we may take $A$ to be a square matrix over $JH$ with nonzero determinant. By the work of Blanchfield [1], $A'$ is also an Alexander matrix of $k$, where $A'$ denotes the transpose of $A$. Let $k'$ denote the knot obtained by inverting $k$, i.e. by reversing the orientation of $S^1$; then $A$, and hence $A'$, is an Alexander matrix of $k'$.

In [3], Fox and Smythe describe two invariants of the knot $k$. If $E$ is the ideal of $JH$ generated by an irreducible factor of the Alexander polynomial $\Delta(t)$, then the row ideal class of $A$ over the ring $JH/E$ is an invariant of $k$. The same is true of the column class.

Denoting the row class by $\rho$ and the column class by $\tau$, we see that the corresponding row class of $k'$ is $\tau$. Thus for $k$ to be invertible it is necessary that $\rho = \tau$; as $\rho \tau = 1$ in the ideal class semigroup (see [5]), this condition reduces to $\rho^2 = 1$.

2. Now let $k$ be the pretzel knot $(25, -3, 13)$, and take $E = (\Delta(t))$. By [3], an Alexander matrix for this knot is

$$A = \begin{bmatrix} 2t - 1 & -5(t - 1) \\ 11(t - 1) & t - 2 \end{bmatrix}.$$

By taking the product of the ideals generated by the first and second rows, and remarking that $t - 1$ is a unit in $JH/E$ (cf. [3]) we find that $\rho^2$ is represented by the ideal $(11(2t-1), 55, 5(t-2))$. By the argument given in

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[3], to prove that this is not a principal ideal it suffices to show that \( J = (55, 11(52 + \sqrt{-211}), 5(107 - \sqrt{-211})/2) \) is not a principal ideal in the ring of integers of the algebraic number field \( \mathbb{Q}(\sqrt{-211}) \).

The h.c.f. of the norms of these generators is 55, and it is easily checked that every element of \( J \) must have norm divisible by 55. Thus if \( J = (\alpha) \), the norm of \( \alpha \) must be 55. Therefore the only possibilities for \( \alpha \) are \((3 \pm \sqrt{-211})/2\): this is a matter of solving the equation \( 4 \cdot 55 = a^2 + 211b^2 \) in integers. Again it is easy to check that \( 11(52 + \sqrt{-211}) \) is not a multiple of \((3 + \sqrt{-211})/2\), nor \( 5(107 - \sqrt{-211})/2 \) of \((3 - \sqrt{-211})/2\).

It follows that \( J \) is not a principal ideal, and so \( \rho^2 \neq 1 \) in the ideal class semigroup.

Therefore \( k \) is not invertible.

3. Theorem 1. There exist noninvertible \( n \)-knots \( S^n \subset S^{n+2} \) for all \( n \geq 1 \).

Proof. Let \( k \) be the pretzel knot \((25, -3, 13)\); and let \( k_n \) be the \( n \)-knot obtained by spinning \( k \) \((n-1)\) times. Then the group of \( k_n \) is isomorphic to that of \( k \), and so the two knots have the same Alexander matrices and hence the same row ideal classes. By the argument above, \( k_n \) is noninvertible. \( \square \)

To emphasize that noninvertibility depends on the module structure rather than on the knot group, we give the following result.

Theorem 2. If \( 1 < q < n/2 \), there exists a noninvertible \( n \)-knot \( S^n \subset S^{n+2} \) with \( \pi_q(S^{n+2} - S^n) \cong \pi_q(S^1) \), \( i < q \).

Proof. By a result of Kervaire [4], there exists such an \( n \)-knot with \( \pi_q(S^{n+2} - S^n) \) presented as a \( JH \)-module by the matrix \( A \) given above. \( \square \)

4. It was pointed out in [2] that a 2-knot with asymmetric Alexander polynomial is noninvertible: by spinning this generalises to \( n > 2 \). Trotter settled the case \( n = 1 \) in [7]. The method above gives a unified proof for all \( n \), and shows that noninvertibility in higher dimensions need not depend on an asymmetric polynomial.

After writing this paper, I learned that Trotter had previously discovered this method of proving that a 1-knot is noninvertible [6].

References


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