ON A PROPERTY OF RATIONAL FUNCTIONS. II

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Abstract. It is shown that if $r_n(z)$ is a rational function of degree $n$ such that $r_n(0) = 1$, $\lim_{|z| \to \infty} |r_n(z)| = 0$ and all its poles lie in $|\zeta_1| \leq |z| \leq 1$ then $\max_{|z|=\rho < |\zeta_1|} |r_n(z)| \geq 1/(1-\rho^n)$.

If $r_n(z)$ is a rational function of $n$th degree

\begin{equation}
\begin{aligned}
&1) \quad r_n(z) = u(z)/v(z), \quad u, v \text{ polynomials} \\
&2) \quad \deg u(z) < \deg v(z) = n, \quad r_n(0) \neq 0
\end{aligned}
\end{equation}

then [1] for $1 < p \leq 2$ and arbitrary complex $z_0$,

\begin{equation}
\|r_n\|_{p, z_0, \rho} = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |r_n(z_0 + \rho e^{i\varphi})|^p \, d\varphi \right)^{1/p}
\end{equation}

can be estimated nontrivially from below in the maximal regularity circle around $z_0$ exclusively by $p, |r_n(z_0)|, \rho, n$ and by the maximum distance of $z_0$ from the poles. Here we obtain the sharp lower bound for

\begin{equation}
\|r_n\|_{\infty, z_0, \rho} = \max_{-\pi \leq \varphi < \pi} |r_n(z_0 + \rho e^{i\varphi})|.
\end{equation}

Without loss of generality we may suppose $z_0 = 0, r_n(0) = 1$; further if the poles are $\zeta_1, \zeta_2, \cdots, \zeta_n$ (repetition permitted), then

\begin{equation}
0 < |\zeta_1| \leq |\zeta_2| \leq \cdots \leq |\zeta_n| = 1,
\end{equation}

and

\begin{equation}
0 < \rho < |\zeta_1|.
\end{equation}

Theorem 1. With the above normalizations we have

\begin{equation}
\max_{|z|=\rho} |r_n(z)| \geq \frac{1}{1 - \rho^n}.
\end{equation}

The example $r_n^*(z) = 1/(1 - z^n)$ shows that (7) is sharp.
Proof of Theorem 1. Let \( \max_{|z|=\rho} |r_n(z)| = M(\rho) \). For every real \( \alpha \) the function \( u(z) - e^{i\alpha}M(\rho)v(z) \) does not vanish in \( |z|<\rho \). Hence if

\[
u(z) = 1 + \sum_{k=1}^{n} b_k z^k,
\]

then

\[
u(\rho z) - e^{i\alpha}M(\rho)v(\rho z) \equiv (1 - e^{i\alpha}M(\rho)) + (a_1 - e^{i\alpha}M(\rho)b_1)(\rho z) + \cdots + (a_{n-1} - e^{i\alpha}M(\rho)b_{n-1})(\rho z)^{n-1} - e^{i\alpha}M(\rho)b_n(\rho z)^n
\]
does not vanish in \( |z|<1 \). Consequently, the coefficient of \( z^n \) does not exceed in absolute value the constant term, i.e. \( |1 - e^{i\alpha}M(\rho)| \geq | - e^{i\alpha}M(\rho)b_n\rho^n| \). Choosing \( \alpha \) appropriately, we get \( M(\rho)(1 - |b_n|\rho^n) \geq 1 \) or \( M(\rho) \geq 1/(1 - |b_n|\rho^n) \). But \( |b_n| \geq 1 \) since the zeros of \( v(z) \) lie in \( |z|\leq 1 \). Hence \( M(\rho) \geq 1/(1 - \rho^n) \).

The condition (2) is automatically satisfied in the important special case when \( r_n(z) \) is the logarithmic derivative of a polynomial \( \pi_n(z) \) of degree \( n \) with all zeros in the unit disk; hence Theorem 1 gives a lower bound for \( \max_{|z|=\rho} |\pi_n'(z)/\pi_n(z)| \) depending on \( |\pi_n'(0)/\pi_n(0)|, \rho, n \). However, we show that in this case there is a lower bound which is independent of \( |\pi_n'(0)/\pi_n(0)| \).

In fact, we prove

Theorem 2. Let the zeros \( \zeta_1, \zeta_2, \ldots, \zeta_n \) of the polynomial \( \pi_n(z) \) of degree \( n \) be such that \( 0 < |\zeta_1| \leq |\zeta_2| \leq \cdots \leq |\zeta_n| \leq 1 \). Then for \( 0 \leq \rho < |\zeta_1| \)

\[
(8) \quad \max_{|z|=\rho} \left| \frac{\pi_n'(z)}{\pi_n(z)} \right| \geq \frac{n\rho^{n-1}}{1 - \rho^n}.
\]

The example \( \pi_n(z) = 1 - z^n \) shows that (8) is sharp.

Proof of Theorem 2. Let \( \pi_n(z) = \sum_{k=0}^{n} c_k z^k \) and set \( A = \max_{|z|=\rho} |\pi_n'(z)/\pi_n(z)| \).

Then

\[
A = \max_{|z|=\rho} \left| \frac{\pi_n'(z)}{\pi_n(z)} \right| = \max_{|z|=1} \left| \frac{c_1 + 2c_2 z + \cdots + nc_n \rho^{n-1} z^{n-1}}{c_0 + c_1 \rho z + c_2 \rho^2 z^2 + \cdots + c_n \rho^n z^n} \right| = \max_{|z|=1} \left| \frac{n\tilde{c}_n \rho^{n-1} + (n-1)\tilde{\xi}_n z + \cdots + 2\tilde{c}_2 \rho z + \tilde{c}_1 z^{n-1}}{c_0 + c_1 \rho z + c_2 \rho^2 z^2 + \cdots + c_n \rho^n z^n} \right| \geq \frac{n\rho^{n-1} \max_{|z|=1} \left| \frac{\tilde{c}_n z^1 + \cdots + 2\tilde{c}_2 \rho z + \tilde{c}_1 z^{n-1}}{c_0 + c_1 \rho z + c_2 \rho^2 z^2 + \cdots + c_n \rho^n z^n} \right|}{1 + \frac{n - 1 \tilde{c}_n}{n \tilde{c}_n \rho} + \cdots + \frac{2 \tilde{c}_2}{n \tilde{c}_n \rho} + \frac{\tilde{c}_1}{\tilde{c}_n \rho}}.
\]
\[ n \left| \frac{c_n}{c_0} \right| \rho^{n-1} \max_{|z|=\rho} \left| 1 + \frac{n-1}{n} \frac{c_{n-1}}{c_n} \left( \frac{z}{\rho^2} \right) + \cdots + \frac{2}{n} \frac{c_2}{c_n} \left( \frac{z}{\rho^2} \right)^{n-2} + \frac{\tilde{c}_1}{\tilde{c}_n} \left( \frac{z}{\rho^2} \right)^{n-1} \right| \]

\[ \geq n \left| \frac{c_n}{c_0} \right| \rho^{n-1} \frac{1}{1 - \rho^n} \]

by Theorem 1. But \(|c_n/c_0| \geq 1\) since all the zeros of \(\pi_n(z)\) lie in \(z \leq 1\). Hence \(A \geq n\rho^{n-1}/(1 - \rho^n)\).

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**REFERENCE**


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