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ON A PROPERTY OF RATIONAL FUNCTIONS. II

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ABSTRACT. It is shown that if $r_n(z)$ is a rational function of degree n such that $r_n(0)=1$, $\lim_{|z|\to\infty}|r_n(z)|=0$ and all its poles lie in $|\zeta_1|\leqq|z|\leqq 1$ then $\max_{|z|=\rho<|\zeta_1|}|r_n(z)|\geqq 1/(1-\rho^n)$.

If $r_n(z)$ is a rational function of nth degree

$$r_n(z) = u(z)/v(z), \qquad u, v \text{ polynomials} \tag{1}$$

with

$$\deg u(z) < \deg v(z) = n, \qquad r_n(0) \neq 0 \tag{2}$$

then [1] for $1<p\leqq 2$ and arbitrary complex z_0,

$$\|r_n\|_{p,z_0,\rho} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|r_n(z_0+\rho e^{i\varphi})|^p\,d\varphi\right)^{1/p} \tag{3}$$

can be estimated nontrivially from below in the maximal regularity circle around z_0 exclusively by p, $|r_n(z_0)|$, ρ, n and by the maximum distance of z_0 from the poles. Here we obtain the sharp lower bound for

$$\|r_n\|_{\infty,z_0,\rho} = \max_{-\pi\leqq\varphi<\pi}|r_n(z_0+\rho e^{i\varphi})|. \tag{4}$$

Without loss of generality we may suppose $z_0=0$, $r_n(0)=1$; further if the poles are $\zeta_1, \zeta_2, \cdots, \zeta_n$ (repetition permitted), then

$$0 < |\zeta_1| \leqq |\zeta_2| \leqq \cdots \leqq |\zeta_n| = 1, \tag{5}$$

and

$$0 < \rho < |\zeta_1|. \tag{6}$$

THEOREM 1. *With the above normalizations we have*

$$\max_{|z|=\rho}|r_n(z)| \geqq \frac{1}{1-\rho^n}. \tag{7}$$

The example $r_n^*(z)=1/(1-z^n)$ shows that (7) is sharp.

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PROOF OF THEOREM 1. Let $\max_{|z|=\rho}|r_n(z)|=M(\rho)$. For every real α the function $u(z)-e^{i\alpha}M(\rho)v(z)$ does not vanish in $|z|<\rho$. Hence if

$$u(z) = 1 + \sum_{k=1}^{n-1} a_k z^k, \qquad v(z) = 1 + \sum_{k=1}^{n} b_k z^k,$$

then

$$u(\rho z) - e^{i\alpha}M(\rho)v(\rho z) \equiv (1 - e^{i\alpha}M(\rho)) + (a_1 - e^{i\alpha}M(\rho)b_1)(\rho z) + \cdots + (a_{n-1} - e^{i\alpha}M(\rho)b_{n-1})(\rho z)^{n-1} - e^{i\alpha}M(\rho)b_n(\rho z)^n$$

does not vanish in $|z|<1$. Consequently, the coefficient of z^n does not exceed in absolute value the constant term, i.e. $|1-e^{i\alpha}M(\rho)|\geqq|-e^{i\alpha}M(\rho)b_n\rho^n|$. Choosing α appropriately, we get $M(\rho)(1-|b_n|\rho^n)\geqq 1$ or $M(\rho)\geqq 1/(1-|b_n|\rho^n)$. But $|b_n|\geqq 1$ since the zeros of $v(z)$ lie in $|z|\leqq 1$. Hence $M(\rho)\geqq 1/(1-\rho^n)$.

The condition (2) is automatically satisfied in the important special case when $r_n(z)$ is the logarithmic derivative of a polynomial $\pi_n(z)$ of degree n with all zeros in the unit disk; hence Theorem 1 gives a lower bound for $\max_{|z|=\rho}|\pi_n'(z)/\pi_n(z)|$ depending on $|\pi_n'(0)/\pi_n(0)|$, ρ, n. However, we show that in this case there is a lower bound which is independent of $|\pi_n'(0)/\pi_n(0)|$. In fact, we prove

THEOREM 2. *Let the zeros $\zeta_1, \zeta_2, \cdots, \zeta_n$ of the polynomial $\pi_n(z)$ of degree n be such that $0<|\zeta_1|\leqq|\zeta_2|\leqq\cdots\leqq|\zeta_n|\leqq 1$. Then for $0\leqq\rho<|\zeta_1|$*

$$\max_{|z|=\rho}\left|\frac{\pi_n'(z)}{\pi_n(z)}\right| \geqq \frac{n\rho^{n-1}}{1-\rho^n}. \tag{8}$$

The example $\pi_n^*(z)=1-z^n$ shows that (8) is sharp.

PROOF OF THEOREM 2. Let $\pi_n(z)=\sum_{k=0}^n c_k z^k$ and set

$$A = \max_{|z|=\rho}|\pi_n'(z)/\pi_n(z)|.$$

Then

$$A = \max_{|z|=1}\left|\frac{\pi_n'(\rho z)}{\pi_n(\rho z)}\right| = \max_{|z|=1}\left|\frac{c_1 + 2c_2\rho z + \cdots + nc_n\rho^{n-1}z^{n-1}}{c_0 + c_1\rho z + c_2\rho^2 z^2 + \cdots + c_n\rho^n z^n}\right|$$

$$= \max_{|z|=1}\left|\frac{n\bar{c}_n\rho^{n-1} + (n-1)\bar{c}_{n-1}\rho^{n-2}z + \cdots + 2\bar{c}_2\rho z^{n-2} + \bar{c}_1 z^{n-1}}{c_0 + c_1\rho z + c_2\rho^2 z^2 + \cdots + c_n\rho^n z^n}\right|$$

$$= n\left|\frac{\bar{c}_n}{c_0}\right|\rho^{n-1}\max_{|z|=1}\left|\frac{1 + \frac{n-1}{n}\frac{\bar{c}_{n-1}}{\bar{c}_n}\frac{z}{\rho} + \cdots + \frac{2}{n}\frac{\bar{c}_2}{\bar{c}_n}\left(\frac{z}{\rho}\right)^{n-2} + \frac{\bar{c}_1}{\bar{c}_n}\left(\frac{z}{\rho}\right)^{n-1}}{1 + \frac{c_1}{c_0}\rho z + \frac{c_2}{c_0}\rho^2 z^2 + \cdots + \frac{c_n}{c_0}\rho^n z^n}\right|$$

$$= n\left|\frac{c_n}{c_0}\right|\rho^{n-1}\max_{|z|=\rho}\left|\frac{1+\dfrac{n-1}{n}\dfrac{\bar{c}_{n-1}}{\bar{c}_n}\left(\dfrac{z}{\rho^2}\right)+\cdots+\dfrac{2}{n}\dfrac{\bar{c}_2}{\bar{c}_n}\left(\dfrac{z}{\rho^2}\right)^{n-2}+\dfrac{\bar{c}_1}{\bar{c}_n}\left(\dfrac{z}{\rho^2}\right)^{n-1}}{1+\dfrac{c_1}{c_0}z+\dfrac{c_2}{c_0}z^2+\cdots+\dfrac{c_n}{c_0}z^n}\right|$$

$$\geqq n\left|\frac{c_n}{c_0}\right|\rho^{n-1}\frac{1}{1-\rho^n}$$

by Theorem 1. But $|c_n/c_0|\geqq 1$ since all the zeros of $\pi_n(z)$ lie in $z\leqq 1$. Hence $A\geqq n\rho^{n-1}/(1-\rho^n)$.

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Reference

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