ISOMETRIC EMBEDDING OF A COMPACT RIEMANNIAN MANIFOLD INTO EUCLIDEAN SPACE

HOWARD JACOBOWITZ

ABSTRACT. An isometric immersion of an n-dimensional compact Riemannian manifold with sectional curvature always less than $\lambda^{-2}$ into Euclidean space of dimension $2n-1$ can never be contained in a ball of radius $\lambda$. This generalizes and includes results of Tompkins and Chern and Kuiper.

Tompkins [4] proved that the flat n-dimensional torus could not be isometrically embedded into $E$, the Euclidean space of dimension $2n-1$. Chern and Kuiper [1] and Otsuki [3] generalized this and showed that a compact n-dimensional manifold whose sectional curvatures are everywhere nonpositive also cannot be isometrically embedded in $E$. Apparently these results also hold for immersions. The purpose of this note is to point out that a standard proof of this nonembeddability result essentially establishes a more general quantitative version.

THEOREM. Let $E$ be Euclidean space of dimension $2n-1$ and $M$ a compact n-dimensional Riemannian manifold whose sectional curvatures are everywhere less than some constant $\lambda^{-2}$. Then no isometric immersion of $M$ into $E$ is contained in a ball of radius $\lambda$.

To prove the theorem, we adapt the proof for nonpositive curvature given in Kobayashi and Nomizu [2, pp. 26–29]. Note the theorem includes the results mentioned above.

Let $f: M \rightarrow E$ be an isometric embedding. Identify the tangent space $T_x M$ for $x \in M$ with its realization as a linear subspace of $E$. Denote the length and inner product of vectors in $E$ by $|X|$ and $(X, Y)$. To prove the theorem, let us assume such an isometric immersion did exist. We can assume $|f(x)| \leq \lambda$ while $|f(x_0)| = \lambda$ for some $x_0$ and all $x$ in $M$. Thus $\langle f(x_0), X \rangle = 0$ for all $X$ in $T_{x_0} M$. Let $L(X, Y)$ denote the second fundamental form at $x_0$ of $M$ in $E$. For $X$ and $Y$ in $T_{x_0} M$, $L(X, Y)$ is a vector in $E$ orthogonal to $M$ at $x_0$. The sectional curvature of the two-plane spanned by $X$ and $Y$ in $M$ is $\frac{1}{|X|^2} L(X, Y)$.
linearly independent vectors $X$ and $Y$ in $T_{x_0} M$ is given by

$$K(X, Y) = \left( |X|^2 |Y|^2 - \langle X, Y \rangle^2 \right)^{-1} \cdot \left( \langle L(X, X), L(Y, Y) \rangle - \langle L(X, Y), L(X, Y) \rangle \right).$$

For a proof of the theorem it suffices to find two vectors $X$ and $Y$ such that $K(X, Y) \geq \lambda^{-2}$.

Differentiating $\langle f(x), f(x) \rangle$ twice at the maximum $x = x_0$, one obtains $\langle L(X, X), v \rangle \leq -\lambda^{-1}|X|^2$ for the unit normal $v = \lambda^{-1} f(x_0)$ and all $X \in T_{x_0} M$. In particular $L(X, X) = 0$ only for $X = 0$. Now for any symmetric bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with $B(X, X) = 0$ only for $X = 0$, one can find linearly independent vectors $X$ and $Y$ such that $B(X, X) = B(Y, Y)$ and $B(X, Y) = 0$. Indeed just solve $B(Z, Z) = 0$ where $Z$ is allowed to be complex. This observation is due to T. A. Springer; see [2]. Let $X$ and $Y$ be two such vectors for $L$. Since $\langle L(X, X), v \rangle \leq -\lambda^{-1}|X|^2$ while $L(X, X) = L(Y, Y)$ one has $\langle L(X, X), L(Y, Y) \rangle \geq \lambda^{-2}|X|^2|Y|^2$. This together with $L(X, Y) = 0$ implies the inequality $K(X, Y) \geq \lambda^{-2}$ and so proves the theorem.

If for the manifold $M$ we only knew that for each $x$, $T_x M$ contained a $q$-dimensional subspace $T'_x$ with the property that the sectional curvature of each two plane in $T'_x$ was greater than $\lambda^{-2}$, then we could conclude that no isometric immersion of $M$ into $E^{n+q-1}$ is contained in a ball of radius $\lambda$. This result, for nonpositive curvatures, is due to Chern and Kuiper [1].

References


Department of Mathematics, Rice University, Houston, Texas 77001