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ON REPRESENTATIONS OF THE GROUP OF LISTING'S KNOT BY SUBGROUPS OF $SL(2, C)$

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ABSTRACT. The subgroups G of $SL(2, C)$ which represent the group of Listing's knot are characterized by the traces of A, B and AB, where A and B are matrices which generate G. From this it follows that there exist finitely generated nondiscrete subgroups of $SL(2, \boldsymbol{R})$ which are not isomorphic to any Fuchsian group.

1. **Introduction and statement of results.** The special linear group $SL(2, C)$, where C is the field of complex numbers, is the group of all 2×2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad ad - bc = 1, \qquad a, b, c, d \in \mathbf{C}. \tag{1.1}$$

For $A \in SL(2, C)$ let $\sigma(A)$ denote the trace of A. The quotient group $SL(2, C)/\{\pm I\}$ is isomorphic to the group Ω of all linear fractional transformations $T(z)=(az+b)/(cz+d)$ of the extended complex plane into itself with complex coefficients and determinant 1. In this paper an element T of Ω is identified with a matrix of the form (1.1), with the understanding that a matrix and its negative define the same T.

Let F_n be a free group on n free generators x_k, $k=1, \cdots, n$; let ρ be a representation of F_n by a subgroup G of $SL(2, C)$. It is known [2], [3] that the characters of all the elements of F_n can be expressed in terms of the characters of a finite subset S_n of F_n consisting of the generators x_k and certain of their products, with $|S_n|=2^n-1$. When ordered, the 2^n-1 complex numbers $\sigma(\rho(u))$, $u \in S_n$, determine an element σ_ρ of C^{2^n-1}, and the character manifold

$$\mathscr{T}_n = \{\sigma_\rho / \rho : F_n \to SL(2, C),\ \rho \text{ is a representation}\}$$

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describes the continuum of all representations of F_n by subgroups of $SL(2, C)$. When $n=2$ and $\rho(x_1)$, $\rho(x_2)$, $\rho(x_1x_2)$ do not all have trace ± 2, the point $\sigma_\rho=(\sigma(\rho(x_1)), \sigma(\rho(x_2)), \sigma(\rho(x_1x_2)))$ of $\mathscr{T}_2$ determines the representation ρ up to conjugation.

The braid group B_n is the group of automorphisms of F_n generated by the automorphisms β_k, $k=1, \cdots, n-1$, with $\beta_k: x_k \to x_{k+1}$, $\beta_k: x_{k+1} \to x_{k+1}x_kx_{k+1}^{-1}$, $\beta_k: x_i \to x_i$, $i \neq k$, $k+1$. The fact that every knot group can be obtained from F_n (for appropriate n) by identifying the generators x_k with their images under an element β of B_n [**1**] has motivated a long standing conjecture of Wilhelm Magnus: the points of the manifold $\mathscr{T}_n$ corresponding to representations of a knot group are the fixed points of the automorphism of $\mathscr{T}_n$ induced by the associated braid automorphism $\beta \in B_n$. The following theorem describes the submanifold of $\mathscr{T}_2$ corresponding to representations of the group of Listing's knot.

THEOREM 1. *Let $L=\langle a, b; R(a, b)\rangle$ with $R(a, b)=b^{-1}a^{-1}bab^{-1}aba^{-1}b^{-1}a$ be a presentation of the group of Listing's knot; let Λ denote the set of all subgroups of $SL(2, C)$ which represent L. An abelian group $G \in \Lambda$ if and only if G is cyclic; a nonabelian group $G \in \Lambda$ if and only if $G=\langle A, B\rangle$, where $\sigma(A)=\sigma(B)$ is an arbitrary complex number x, and*

$$\sigma(AB) = \tfrac{1}{2}(1 + x^2 \pm ((x^2 - 1)(x^2 - 5))^{1/2}). \tag{1.2}$$

The question of faithfulness of the representations given by the theorem is open. The derived group L' of L is free of rank 2 [**5**], and a group G in Λ will faithfully represent L if and only if G' is free of rank 2, but there are presently no known criteria for determining the freeness of subgroups of the groups in Λ.

For $x \in C$ let $G_x=\langle A_x, B_x\rangle$ be a nonabelian group in Λ with $\sigma(A_x)=\sigma(B_x)=x$, and $\sigma(A_xB_x)$ given by (1.2). We may conjugate so that $G_{\pm 2}$ is generated by

$$\pm\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \pm\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}, \qquad t = e^{i\pi/3},$$

and for $x \neq \pm 2$, G_x is generated by

$$\begin{aligned} A_x &= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \qquad \lambda = \tfrac{1}{2}(x \pm (x^2 - 4)^{1/2}, \\ B_x &= \begin{bmatrix} \mu & 1 \\ \mu(x-\mu) - 1 & x - \mu \end{bmatrix} \end{aligned} \tag{1.3}$$

$\mu=(\lambda z-x)/(\lambda^2-1)$, where $z=\sigma(AB)$ is given by (1.2). If G_x is regarded as a subgroup of Ω, then $A_{-x}=A_x^{-1}$, $B_{-x}=B_x^{-1}$; hence $G_x=G_{-x}$. $G_{5^{1/2}}$ is

torsion free metabelian, G_1 is generated by elliptic transformations of order 3 and is not discrete [**4**, Theorem 2.3]. For x real, $x^2>5$, G_x is not metabelian; moreover the following theorem holds.

THEOREM 2. *Let* $z=\frac{1}{2}(1+x^2+((x^2-1)(x^2-5))^{1/2})$ *where* x *is real and transcendental with* $x^2>5$. *Let* $G=\langle A, B\rangle$ *where* $A=A_x$, $B=B_x$ *are given by* (1.3). G *is a finitely generated nondiscrete subgroup of* $SL(2, \boldsymbol{R})$ *which is not isomorphic to any Fuchsian group.*

A more detailed description of the nonabelian groups in Λ awaits further investigation.

2. **The proof of the theorems.** We need the following facts about the trace function on $SL(2, \boldsymbol{C})$ [**2**]:

For all $A, B, C \in SL(2, \boldsymbol{C})$

(2.1) $\sigma(A)=\sigma(A^{-1})$,

(2.2) $\sigma(AB)=\sigma(A)\cdot\sigma(B)-\sigma(AB^{-1})$,

(2.3) $\sigma(ABC)=\sigma(A)\cdot\sigma(BC)+\sigma(B)\cdot\sigma(AC)+\sigma(C)\cdot\sigma(AB)-\sigma(A)\cdot\sigma(B)\sigma(C)-\sigma(ACB)$.

When applying (2.3) to rewrite $\sigma(X_1X_2\cdots X_n)$, $X_i\in SL(2, \boldsymbol{C})$, $i=1,\cdots,n$, we say that $\sigma(X_1X_2\cdots X_n)$ is expanded according to the subdivision $X_1\cdots X_k(X_{k+1}\cdots X_m)X_{m+1}\cdots X_n$ if $A=X_1\cdots X_k$, $B=X_{k+1}\cdots X_m$, $C=X_{m+1}\cdots X_n$.

The proof of Theorem 1 requires the following two lemmas.

LEMMA 1. *Let* $A, B\in SL(2, \boldsymbol{C})$; *let* $R=B^{-1}A^{-1}BAB^{-1}ABA^{-1}B^{-1}A$ *and suppose* $\sigma(A)=\sigma(B)=x$, $\sigma(AB)=z$. *Let* $f=f(x, z)=x^2-z-2$, $g=g(x, z)=z^2-(1+x^2)z+2x^2-1$. *Then*

(i) $\sigma(RB)=x$,

(ii) $\sigma(R)-2=fg^2$,

(iii) $\sigma(RA^{-1})-x=x(2-z)fg$.

PROOF. (i) follows from the fact that RB and A are conjugate in $SL(2, \boldsymbol{C})$. Let $C(x, z)=\sigma(B^{-1}A^{-1}BA)=2x^2+z^2-x^2z-2$. Expansion of $\sigma(R)$ via (2.3) according to the subdivision $B^{-1}A^{-1}BA(B^{-1}AB)A^{-1}B^{-1}A$, using (2.1), (2.2) and the fact that the trace function is invariant under conjugation, yields $\sigma(R)-2=x^2(C-1)^2-z(C^2-C-1)-2=fg^2$.

Expansion of $\sigma(RA^{-1})$ according to the subdivision

$$B^{-1}ABA(B^{-1}AB)A^{-1}B^{-1}$$

yields $\sigma(RA^{-1})-x=x(z(2-C)+C(C-1)-2)=x(2-z)fg$.

LEMMA 2. *Let* $A, B, R\in SL(2, \boldsymbol{C})$ *with* $\sigma(A)=\sigma(B)=x$ *and* $\sigma(AB)\neq 2$, $\sigma(AB)\neq x^2-2$. *If* R *satisfies* $\sigma(R)=2$, $\sigma(RA)=\sigma(RB)=x$, *then* $R=I$.

PROOF. If $R \neq I$ then we may conjugate A, B and R so that $R=\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. $\sigma(RA)=\sigma(RB)=x$ implies that

$$A = \begin{pmatrix} \alpha & \gamma \\ 0 & \alpha^{-1} \end{pmatrix} \quad \text{for } \alpha = \tfrac{1}{2}(x \pm (x^2 - 4)^{1/2}),$$

and

$$B^{\pm 1} = \begin{pmatrix} \alpha & \mu \\ 0 & \alpha^{-1} \end{pmatrix}, \qquad \gamma, \mu \in \boldsymbol{C}.$$

Hence $\sigma(AB)=x^2-2$ or $\sigma(AB)=2$.

PROOF OF THEOREM 1. Since L/L' is infinite cyclic [5] it is clear that an abelian subgroup G of $SL(2, \boldsymbol{C})$ belongs to Λ if and only if G is cyclic. If $G \in \Lambda$ is nonabelian and $\rho: L \to G$ is an epimorphism with $\rho(a)=A$, $\rho(b)=B$, then since A and B are conjugate in G, $\sigma(A)=\sigma(B)$. Without loss of generality A is in Jordan canonical form. Using the notation of Lemma 1, by (ii) either $f(x, z)=0$ or $g(x, z)=0$. In the latter case (1.2) of the theorem is satisfied. If $f(x, z)=0$ then $x \neq \pm 2$, else G would be abelian, contrary to assumption. Thus we may write

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda + \frac{1}{\lambda} = x, \qquad B = \begin{pmatrix} \mu & \beta \\ \gamma & x - \mu \end{pmatrix}, \mu(x - \mu) - \beta\gamma = 1.$$

Setting $\sigma(AB)=x^2-2$ and solving for μ yields $\mu=\lambda$ and $B=\left(\begin{smallmatrix}\lambda & \beta\\ \gamma & \lambda^{-1}\end{smallmatrix}\right)$ with either β or γ equal to 0. The relation $R=I$ forces $\lambda^4-3\lambda^2+1=0$; hence $x=\lambda+1/\lambda=5^{1/2}$, $\sigma(AB)=3$, and x, $z=\sigma(AB)$ again satisfy (1.2).

Conversely, let $G=\langle A, B\rangle \in SL(2, \boldsymbol{C})$ with $\sigma(A)=\sigma(B)=x$, $\sigma(AB)=\frac{1}{2}(1+x^2 \pm ((x^2-1)(x^2-5))^{1/2})$. Then $\sigma(AB)$ is different from 2 and x^2-2, whence G is nonabelian. Since $g(x, z)=0$, by Lemma 1 $\sigma(R)=2$, $\sigma(RB)=x$ and $\sigma(RA^{-1})=x$; hence by (2.2) $\sigma(RA)=x$. Lemma 2 now applies to complete the proof.

PROOF OF THEOREM 2. For $x^2>5$ and transcendental,

$$z = \tfrac{1}{2}(1 + x^2 + ((x^2 - 1)(x^2 - 5))^{1/2}),$$

and A and B given by (1.3), G is clearly a subgroup of $SL(2, \boldsymbol{R})$. By Theorem 1 $G \in \Lambda$. By (2.2) $\sigma(BAB^{-2})=x^2-z$. The choices of x and z imply that BAB^{-2} is elliptic of infinite order, hence G is not discrete.

It is known [7] that a Fuchsian group of rank 2 has a presentation of one of the following three types:

(i) $H_{k,l,m}=\langle C, D; C^k, D^l, (CD)^m\rangle$ with $1/k+1/l+1/m<1$;

(ii) $H_k=\langle C, D; [C, D]^k\rangle$;

(iii) $H_{k,l}=\langle C, D; C^k, D^l\rangle$.

The relations in $H_{k,l,m}$, and the fact that the traces of all group elements are polynomials with integer coefficients in $\sigma(C)$, $\sigma(D)$, $\sigma(CD)$ [3], imply

that the traces of all elements in $H_{k,l,m}$ are algebraic numbers. Thus G cannot be isomorphic to a group of type (i). In case (ii), if $\varphi: H_k \to G$ were an isomorphism, then $[A, B]$ would be conjugate in G to $\varphi([C, D]^{\pm 1})$ [**6**, Theorem 4], forcing $\sigma([A, B])$ to be algebraic. But the relation $g(x, z) = 0$ satisfied by x and z implies that $\sigma([A, B]) = z - 1$, where z is transcendental. To dispense with case (iii), one needs the facts, implicit in the results of [**5**], that the derived group L' of L is free on two generators, and that G is isomorphic to L if G' is also free of rank 2. For the derived group of $H_{k,l}$ is free of rank $(k-1)(l-1)$; thus the assumption that G is isomorphic to $H_{k,l}$ implies that $(k, l) = (2, 3)$ and that G' is free of rank 2. But then G, and hence the group $H_{2,3}$, would be isomorphic to the torsion free group L, which is impossible.

This completes the proof of Theorem 2.

References

1. E. Artin, *Theorie der Zöpfe*, Abh. Math. Sem. Univ. Hamburg **4** (1926), 47–72.

2. R. Fricke and F. Klein, *Vorlesungen über die Theorie der automorphen Functionen*, Vol. 1, Teubner, Leipzig, 1897, pp. 338, 366.

3. R. Horowitz, *Characters of free groups represented in the two-dimensional special linear group*, Comm. Pure Appl. Math. **25** (1972), 635–650.

4. A. W. Knapp, *Doubly generated Fuchsian groups*, Michigan Math. J. **15** (1968), 289–304. MR **40** #1483.

5. W. Magnus, *Untersuchungen über einige unendliche diskontinuierliche Gruppen*, Math. Ann. **105** (1931), 52–74.

6. G. Rosenberger, *Automorphismen und Erzeugende fur Gruppen mit einer definierenden Relation*, Math. Z. **129** (1972), 259–267.

7. H. Zieschang, Unpublished.

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