

COMMUTATIVE EXTENSIONS BY CANONICAL MODULES ARE GORENSTEIN RINGS

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ABSTRACT. Reiten has demonstrated that the trivial Hochschild extension of a Cohen-Macaulay local ring by a canonical module is a Gorenstein local ring. Here it is proved that any commutative extension of a Cohen-Macaulay local ring by a canonical module is a Gorenstein ring. Also Gorenstein extensions of a local Cohen-Macaulay ring by a module are studied.

Introduction. Suppose A is a commutative ring and that M is an A -module. A *commutative extension* of A by M is an exact sequence of abelian groups

$$0 \longrightarrow M \xrightarrow{i} E \xrightarrow{\pi} A \longrightarrow 0,$$

where E is a commutative ring, the map π is a ring homomorphism and the A -module structure on M is related to (E, i, π) by the equations

$$ei(x) = i(\pi(e)x), \quad \text{for all } e \in E \text{ and all } x \in M.$$

The i identifies M with an ideal of square zero in E . (On the other hand if \mathfrak{J} is an ideal of square zero in E , then \mathfrak{J} is an E/\mathfrak{J} -module and $0 \rightarrow \mathfrak{J} \rightarrow E \rightarrow E/\mathfrak{J} \rightarrow 0$ is an extension of E/\mathfrak{J} by \mathfrak{J} .)

The *trivial extension* of A by M is the exact sequence

$$0 \longrightarrow M \xrightarrow{i} M \times A \xrightarrow{\pi} A \longrightarrow 0$$

where i is the first coordinate map, where π is the second projection and where $M \times A$ is a ring whose underlying additive structure is the direct sum of abelian groups and whose multiplication is given elementwise by $(m, a)(m', a') = (ma' + m'a, aa')$ for all $m, m' \in M$ and all $a, a' \in A$. This extension is denoted by $A \bowtie M$.

Now suppose A is a commutative noetherian local ring with maximal ideal m . An A -module of finite type M is a *canonical module* if it has the

Received by the editors September 30, 1972.

AMS (MOS) subject classifications (1970). Primary 13H10.

Key words and phrases. Noetherian, Gorenstein, Cohen-Macaulay, local ring, canonical module, extension.

¹ This research was partially supported by the NSF. I am grateful for many stimulating conversations with my colleague, P. A. Griffith.

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three properties:

- (i) The natural homomorphism $A \rightarrow \text{End}_A(M)$ is a bijection.
- (ii) The groups $\text{Ext}_A^i(M, M) = 0$ for all $i > 0$.
- (iii) The injective dimension $\text{id}_A M < \infty$. If B is a Gorenstein local ring (see Bass [1]) and if $B \rightarrow A$ is a surjection then A is a Cohen-Macaulay ring if and only if $\text{Ext}_B^i(A, B) = 0$ for $i \neq \dim B - \dim A$. If A is Cohen-Macaulay and if $d = \dim B - \dim A$, then $\text{Ext}_B^d(A, B)$ is a canonical A -module (see Grothendieck [5] and Sharp [8]). Furthermore A is Gorenstein if and only if $A \cong \text{Ext}_B^d(A, B)$. As a converse to this result Reiten [7] shows that if the Cohen-Macaulay local ring A has a canonical module M , then the trivial extension $A \times M$ is a Gorenstein ring. A more precise statement is found in Fossum, Griffith and Reiten [3]: The ring $A \times M$ is a Gorenstein ring if and only if the A -module M is a canonical module.

For further properties of canonical modules we refer to Sharp [8], Herzog and Kunz [6], Foxby [4] and Fossum, Griffith and Reiten [3] although the only results which are really needed are:

- (a) If A has a canonical module M , then A is Cohen-Macaulay.
- (b) If M is a canonical A -module and if $a \in A$, then a is regular on M if and only if it is regular on A .
- (c) If M is a canonical A -module and if a is a regular element in A , then M/aM is a canonical A/aA -module. Also we note that when $B \rightarrow A$ is a surjective ring homomorphism, when B is Gorenstein and when A is Cohen-Macaulay then the spectral sequence (Cartan and Eilenberg [2]) with $E_2^{pq} = \text{Ext}_A^p(X, \text{Ext}_B^q(A, B))$ and abutment $\text{Ext}_B^p(X, B)$ degenerates to natural isomorphisms $\text{Ext}_A^p(X, \text{Ext}_B^d(A, B)) \cong \text{Ext}_B^{p+d}(X, B)$ for all A -modules X , for all integers $p \geq 0$ and for $d = \dim B - \dim A$.

(The term canonical module seems to have been introduced by Herzog and Kunz [6]. A more geometric term is *module of dualizing differentials*, which is very suggestive terminology but also rather cumbersome. Sharp [8] uses the term Gorenstein module of rank one.)

Arbitrary extensions. The main result of this paper is the generalization of Reiten's result.

THEOREM. *Suppose A is a local noetherian ring and M is a canonical A -module. If $0 \rightarrow M \rightarrow E \rightarrow A \rightarrow 0$ is a commutative extension of A by M , then E is a Gorenstein ring.*

The proof is based on two lemmas. The first allows a reduction to the Artin case. The second handles the Artin case.

LEMMA 1. *Suppose A , M and E are as in the statement of the theorem. An element e in E is regular if and only if $\pi(e)$ is regular in A .*

PROOF. Since M is a canonical module, an element $\pi(e)$ is regular in A if and only if it is regular on M . If $e \in E$ and if e is regular then e is regular on $i(M)$. But the restriction of e to $i(M)$ is the action of $\pi(e)$ on M . Hence $\pi(e)$ is regular in A .

Suppose, on the other hand, that $\pi(e)$ is regular in A . If $e \cdot x=0$ for some $x \in E$, then $\pi(e) \cdot \pi(x)=0$. Hence $\pi(x)=0$, since $\pi(e)$ is regular in A . There is then an element $m \in M$ such that $x=i(m)$. Then $ex=i(\pi(e)m)$. Since $\pi(e)$ is regular on M and since i is an injection, the element $m=0$. Hence $x=0$. So e is regular in E . Q.E.D.

LEMMA 2. Suppose A is an Artin ring, that M is an A -module with $\text{Ann}_A M=(0)$ and that E is an extension of A by M . Then $i(\text{Socle}(M))=\text{Socle}(E)$.

PROOF. Let \mathfrak{n} be the radical of E and \mathfrak{m} the radical of A (so that $\mathfrak{m}=\pi(\mathfrak{n})$). Now $e \in \text{Socle}(E)$ if and only if $\mathfrak{n}e=(0)$. If $\mathfrak{n}e=(0)$, then $i(M)e=(0)$. But $i(M)e=i(M\pi(e))$. Hence $\mathfrak{n}e=(0)$ implies $M\pi(e)=(0)$. But $\text{Ann}_A M=(0)$ implies that $\pi(e)=0$. Hence $e \in \text{Socle}(E)$ implies $e \in i(M)$, say $e=i(m)$ for some $m \in M$. Now $\mathfrak{n} \cdot i(m)=i(\mathfrak{n} \cdot m)$ so $\mathfrak{m} \cdot m=(0)$. Hence $m \in \text{Socle}(M)$ and then $e \in i(\text{Socle}(M))$. Clearly $i(\text{Socle}(M)) \subseteq \text{Socle}(E)$. Q.E.D.

Now these two lemmas are used to prove the theorem. The proof is by induction on $\dim A$. We know that $\dim A=\dim E$ since $i(M)$ is nilpotent. By Lemma 1 we can conclude that $\text{depth } A=\text{depth } E$. For suppose e is a regular nonunit in E . Then $\pi(e)$ is a regular nonunit in A . Multiplication by e induces the commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \xrightarrow{i} & E & \xrightarrow{\pi} & A \longrightarrow 0 \\
 & & \downarrow \pi(e) & & \downarrow e & & \downarrow \pi(e) \\
 0 & \longrightarrow & M & \xrightarrow{i} & E & \xrightarrow{\pi} & A \longrightarrow 0.
 \end{array}$$

We conclude that the sequence

$$0 \longrightarrow M/\pi(e)M \xrightarrow{i'} E/eE \xrightarrow{\pi'} A/\pi(e)A \longrightarrow 0$$

is exact and is an extension of the local ring $A/\pi(e)A$ by the canonical module $M/\pi(e)M$. Now $\text{depth } E=1+\text{depth } E/eE$. By induction $\text{depth } E/eE=\text{depth } A/\pi(e)A=-1+\text{depth } A$. Hence $\text{depth } E=\text{depth } A$.

But we can also use this reduction to show that E is Gorenstein. For E is Gorenstein if and only if E/eE is Gorenstein.

If $\dim A=0$, then the canonical module $M \cong E(A/\mathfrak{m})$, the injective

envelope of the residue class field of A . Hence $\text{Ann}_A M = (0)$ since $A \cong \text{End}_A(M)$ and $\text{Socle}(M) \cong A/\mathfrak{m}$. By Lemma 2 we get $\text{Socle}(E) \cong E/\mathfrak{m} \cong A/\mathfrak{m}$. Hence E has a simple socle and is therefore Gorenstein (Bass [1]).

Suppose, for inductive purposes, that E' is Gorenstein for $\dim E' < \dim E$ whenever E' is an extension of a local ring by a canonical module. Then we conclude that E is Gorenstein since E/eE is Gorenstein. Q.E.D.

Gorenstein extensions. Now our attention is directed to the converse problem. Suppose E is an extension of A by M , that E is Gorenstein and that A is Cohen-Macaulay. What is the relationship between M and the canonical module $\text{Hom}_E(A, E)$ of A ? (Note that there are Gorenstein extensions of non-Cohen-Macaulay rings. If k is a field, X and Y indeterminates, then $E = k[[X, Y]]/(X^2)$ is Gorenstein. The ideal generated by the image XY is nilpotent. But $k[[X, Y]]/(X^2, XY)$ is not Cohen-Macaulay.)

Suppose E is a Gorenstein local ring and \mathfrak{J} is an ideal of square zero. Let $A = E/\mathfrak{J}$. We assume that A is Cohen-Macaulay.

Let B be a Gorenstein local ring with $\dim B > 0$. Let t be a regular element in B . Then $B/t^n B$ is Gorenstein for all $n \geq 1$. Let $E = B/t^n B$ and let $\mathfrak{J} = t^{n-1}B/t^n B$. Then $E/\mathfrak{J} = B/t^{n-1}B$ and $\mathfrak{J}^2 = (0)$. Now \mathfrak{J} is a canonical E/\mathfrak{J} -ideal if and only if $n = 2$. Hence it is not the case that \mathfrak{J} is the canonical module.

Let $\Omega = \text{Hom}_E(A, E)$ be the canonical module. Then $\Omega \cong \{e \in E : e\mathfrak{J} = (0)\}$. Since $\mathfrak{J}^2 = (0)$, we have $\mathfrak{J} \subseteq \Omega$. We get the commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \mathfrak{a} & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & \mathfrak{J} & \longrightarrow & E & \xrightarrow{\pi} & A \longrightarrow 0 \\
 & & \downarrow & & \downarrow = & & \downarrow \gamma \\
 0 & \longrightarrow & \Omega & \longrightarrow & E & \xrightarrow{\pi'} & \text{Hom}_E(\mathfrak{J}, E).
 \end{array}$$

The homomorphism $\gamma : A \rightarrow \text{Hom}_E(\mathfrak{J}, E)$ is the composition $A \rightarrow \text{Hom}_E(\mathfrak{J}, \mathfrak{J}) \rightarrow \text{Hom}_E(\mathfrak{J}, E)$. Thus $\text{Ker } \gamma = \text{Ann}_A \mathfrak{J}$.

LEMMA 3. *The natural map $A \rightarrow \text{Hom}_E(\mathfrak{J}, \mathfrak{J})$ is a surjection and $\text{Hom}_E(\mathfrak{J}, \mathfrak{J}) \rightarrow \text{Hom}_E(\mathfrak{J}, E)$ is a bijection. Thus $\text{Hom}_E(\mathfrak{J}, \mathfrak{J})$ is identified with both $A/\text{Ann}_A \mathfrak{J}$ and E/Ω .*

PROOF. Since $\mathfrak{J}^2 = (0)$ and since A is Cohen-Macaulay, the group $\text{Ext}_E^i(A, E) = (0)$ for $i > 0$. Hence π' is a surjection. Thus γ is also a surjection. Since $\text{Hom}_E(\mathfrak{J}, \mathfrak{J}) \rightarrow \text{Hom}_E(\mathfrak{J}, E)$ is an injection, the statements of the lemma follow. Q.E.D.

We identify $\text{Coker}(\mathfrak{J} \rightarrow \Omega)$ with \mathfrak{a} , the annihilator of \mathfrak{J} . The ring $\text{Hom}_E(\mathfrak{J}, \mathfrak{J}) \cong \text{Hom}_A(\mathfrak{J}, \mathfrak{J})$ and is denoted by A' . Then the diagram can be displayed as

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \mathfrak{a} & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & \mathfrak{J} & \longrightarrow & E & \xrightarrow{\pi} & A \longrightarrow 0 \\
 & & \downarrow & & \downarrow = & & \downarrow \gamma \\
 0 & \longrightarrow & \Omega & \longrightarrow & E & \xrightarrow{\pi'} & A' \longrightarrow 0 \\
 & & \downarrow & & & & \downarrow \\
 & & \mathfrak{a} & & & & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

LEMMA 4. *The ring A' is Cohen-Macaulay and $\dim A = \dim A'$.*

PROOF. It is enough to prove that $\text{Ext}_E^i(A', E) \cong (0)$ for $i > 0$. The exact sequence $0 \rightarrow \Omega \rightarrow E \rightarrow A' \rightarrow 0$ gives rise to the exact sequence

$$0 \rightarrow \text{Hom}_E(A', E) \rightarrow E \rightarrow \text{Hom}_E(\Omega, E) \rightarrow \text{Ext}_E^1(A', E) \rightarrow 0$$

and the isomorphisms $\text{Ext}_E^i(\Omega, E) \cong \text{Ext}_E^{i+1}(A', E)$ for $i > 0$. Since A is Cohen-Macaulay we get natural isomorphisms $\text{Ext}_E^i(\Omega, E) \cong \text{Ext}_A^i(\Omega, \Omega)$ for all i . Hence $\text{Ext}_E^i(\Omega, E) = 0$ for all $i > 0$ while

$$\text{Hom}_E(\Omega, E) \cong \text{Hom}_A(\Omega, \Omega) \cong A,$$

since Ω is a canonical A -module. (This follows from the remarks in the introduction.) Thus $E \rightarrow \text{Hom}_E(\Omega, E)$ is just $\pi: E \rightarrow A$ which is a surjection. So $\text{Ext}_E^i(A', E) = (0)$ for all $i > 0$ while the canonical A' -module is $\text{Ker } \pi = \mathfrak{J} \cong \text{Hom}_E(A', E) \cong \{e \in E: e\Omega = 0\}$. Q.E.D.

We record several other consequences. We retain the hypotheses and notation.

PROPOSITION. (a) *The sequences*

$$0 \longrightarrow \mathfrak{J} \longrightarrow E \xrightarrow{\pi} A \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \Omega \longrightarrow E \xrightarrow{\pi'} A' \longrightarrow 0$$

are $\text{Hom}_E(-, E)$ dual.

(b) *The sequences $0 \rightarrow \mathfrak{a} \rightarrow A \rightarrow A' \rightarrow 0$ and $0 \rightarrow \mathfrak{J} \rightarrow \Omega \rightarrow \mathfrak{a} \rightarrow 0$ are $\text{Hom}_A(-, \Omega)$ (which is $\text{Hom}_E(-, E)$) dual.*

(c) *The group $\text{Ext}_E^i(\mathfrak{a}, E) = (0)$ for all $i > 0$.*

(d) The A -module \mathfrak{J} is a canonical A' -module while Ω is a canonical A -module. Thus, in particular, the square zero ideal \mathfrak{J} is a canonical A -module if and only if $\text{Ann}_A \mathfrak{J} = (0)$.

(e) We have the following natural isomorphisms:

$$\begin{aligned} A &= \text{Hom}_A(\Omega, \Omega), \\ A' &= \text{Hom}_A(\mathfrak{J}, \mathfrak{J}) \cong \text{Hom}_E(\mathfrak{J}, E) \cong \text{Hom}_A(\mathfrak{J}, \Omega), \\ \mathfrak{J} &= \text{Hom}_A(A', \Omega) \cong \text{Hom}_E(A', E), \text{ and} \\ \Omega &= \text{Hom}_A(A, \Omega) \cong \text{Hom}_E(A, E). \quad \text{Q.E.D.} \end{aligned}$$

Note that Ω is a nilpotent ideal in E which has square zero if and only if $\Omega = \mathfrak{J}$.

Final remarks. A result which is implicit in this article, and which follows immediately is a slight generalization of the Grothendieck-Bass-Sharp result about Cohen-Macaulay factor rings of Gorenstein rings. We state it here for possible future reference.

PROPOSITION. *Suppose A is a local Cohen-Macaulay ring with a canonical module Ω . The factor ring A' of A is Cohen-Macaulay if and only if $\text{Ext}_A^i(A', \Omega) = (0)$ for all $i \neq \dim A - \dim A'$. If A' is Cohen-Macaulay, then it has a canonical module which is just the nonzero Ext group. Q.E.D.*

It should be mentioned that a nontrivial extension has been exhibited at the beginning of this section. Others will arise from symmetric 2-cocycles $f: A \times A \rightarrow \Omega$ which are not coboundaries (if they exist).

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