ON SEMIPRIME P.I. RINGS
WALLACE S. MARTINDALE III

Abstract. The main results proved in this paper are that if $R$ is a semiprime ring satisfying a polynomial identity then (1) the maximal right quotient ring of $R$ is also P.I. and (2) every essential one-sided ideal of $R$ contains an essential two-sided ideal of $R$.

The primary aim of this paper is to settle in the affirmative two recent conjectures of J. W. Fisher [1]: If $R$ is a semiprime ring satisfying a polynomial identity then

(I) the maximal right quotient ring of $R$ satisfies a polynomial identity,

(II) the maximal right quotient ring of $R$ coincides with the maximal left quotient ring of $R$.

The main tool we use is a result (Theorem A, below) just recently proved by L. Rowen [4, Theorem 2], which in turn depends heavily on a fundamental theorem on central polynomials in matrix rings due to E. Formanek [2].

Let $R$ be a ring, not necessarily having a unit element. If $S$ is a subset of $R$, we let $r(S)$ denote $\{x \in R \mid Sx = 0\}$ and $l(S) = \{x \in R \mid xS = 0\}$. A right (left, two-sided) ideal $J$ of $R$ is essential if, for any right (left, two-sided) ideal $K$, $J \cap K = 0$ implies $K = 0$. A ring is semiprime if it has no nonzero nilpotent ideals. It is clear that in a semiprime ring a two-sided ideal $U$ is essential if and only if $r(U) = 0$ or $l(U) = 0$, and an essential two-sided ideal is also essential as a one-sided ideal. Also in a semiprime ring it is easy to see that if $J$ is an essential right ideal then $r(J) = 0$.

The set $Z(R) = \{x \in R \mid xJ = 0 \text{ for some essential right ideal } J\}$ is a two-sided ideal of $R$, called the right singular ideal of $R$. Similarly the left singular ideal $Z'(R)$ is defined. As we shall be concerned with rings $R$ for which $Z(R) = 0$ (see Theorem I) the maximal right quotient ring $Q$ of $R$ may be characterized by the following properties:

(a) $R$ is a subring of $Q$.
(b) If $f \in \text{Hom}_R(J, R)$, where $J$ is an essential right ideal of $R$, then there exists $q \in Q$ such that $qx = f(x)$ for all $x \in J$.

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(c) If \( q \in \mathcal{Q} \), there is an essential right ideal \( J \) of \( R \) such that \( qJ \subseteq R \).

(d) For all \( q \in \mathcal{Q} \), \( q = 0 \) if and only if \( qJ = 0 \) for some essential right ideal \( J \) in \( R \).

It is well known that \( \mathcal{Q} \) is regular in the sense of von Neumann.

As in [1] a ring \( R \) is said to satisfy a polynomial identity if there exists a (homogeneous multilinear) polynomial

\[
f(x_1, x_2, \ldots, x_n) = \sum_{i} \omega_i x_{i_1} x_{i_2} \cdots x_{i_n}
\]

in noncommuting indeterminates \( \{x_i\} \) (where \( i \) ranges over the symmetric group \( S_n \) and \( \omega_i \) lies in the centroid of \( R \)) such that \( f(r_1, r_2, \ldots, r_n) = 0 \) for all \( r_1, r_2, \ldots, r_n \in R \) and the kernel of \( \omega_i = 0 \). Such a ring is called a P.I. ring. We remark that a more general definition of polynomial identity could have been made which does not require \( f \) to be homogeneous multilinear, but the usual linearization process could be used to replace \( f \) by a homogeneous multilinear polynomial of degree not exceeding the degree of \( f \). On the other hand, in case \( R \) is a semiprime P.I. ring, it is well known that \( R \) in fact satisfies the so-called standard identity

\[
\sum_{i} \pm x_{i_1} x_{i_2} \cdots x_{i_n} = 0.
\]

The key theorem which lies behind the results in this paper is due to L. Rowen [4, Theorem 2] and we quote it as

**Theorem A (Rowen).** Let \( R \) be a semiprime P.I. ring with center \( C \) and let \( U \) be a nonzero ideal of \( R \). Then \( U \cap C \neq 0 \).

It is important to observe that the assumption made in [4] that \( R \) has a unit element is superfluous—the same proof may still be used.

**Lemma 1.** Let \( R \) be a semiprime ring with center \( C \) and let \( J \) be a right ideal of \( R \). Then the center of the ring \( J \) is equal to \( J \cap C \).

**Proof.** Let \( a \) lie in the center of \( J \) and let \( x, r \in R \). Then

\[
(ax - xa)(ax - xa) = (axr)ax - (axrx)a + x(axr)a - x(ar)ax
\]

\[
= a(axr)x - a(axrx) + xa(axr) - xa(ar)x
\]

\[
= 0.
\]

Since \( R \) is semiprime, \( ax - xa = 0 \), i.e., \( a \in C \).

Henceforth \( R \) will denote a semiprime P.I. ring with center \( C \). As a corollary to Theorem A we obtain immediately the following result due to Fisher [1, Theorem 1].

**Theorem 1 (Fisher).** \( Z(R) = 0 = Z'(R) \).

**Proof.** If \( Z(R) \neq 0 \), pick \( \lambda \neq 0 \in Z(R) \cap C \) by Theorem A. Then, for some essential right ideal \( J \), \( J\lambda = \lambda J = 0 \). Since \( R \) is semiprime, this contradicts \( r(J) = 0 \).
Lemma 2. If J is an essential right (left) ideal of R, then J is itself a semiprime P.I. ring.

Proof. Suppose \( A^2 = 0 \), where \( A \) is an ideal of J. Then \( AJ \) is a right ideal of \( R \) and \( (AJ)^2 \subseteq A^2 = 0 \). Hence \( AJ = 0 \) since \( R \) is semiprime. But then \( A = 0 \) by Theorem 1.

We are now in a position to settle conjecture I.

Theorem 2. If \( R \) is a semiprime P.I. ring (satisfying the identity \( f \)) then the maximal right quotient ring \( Q \) of \( R \) satisfies this same polynomial identity.

Proof. Write \( f(x_1, x_2, \ldots, x_n) = \sum_{i} \omega_i x_i x_{i_2} \cdots x_{i_n} \), pick \( q_1, q_2, \ldots, q_n \in Q \), and set \( q = f(q_1, q_2, \ldots, q_n) \). By taking finite intersections there exists an essential right ideal \( J \) of \( R \) such that \( q_1J, q_2J, \ldots, q_nJ, qJ \) are all contained in \( R \). Let \( a \in J \) and write \( qa = b \in R \). Suppose \( b \neq 0 \). Then \( U = RbR \cap J \neq 0 \), since \( J \) is essential, and \( U \) is a two-sided ideal of the ring \( J \). By Lemma 2, \( J \) is a semiprime P.I. ring, and so we can apply Theorem A to \( J \) to conclude that \( U \) contains a nonzero element \( \lambda \) in the center of \( J \). But then by Lemma 1 \( \lambda \in C \). Therefore

\[
b\lambda^n = qa\lambda^n = \sum_i \omega_i q_{i_1}q_{i_2} \cdots q_{i_n}a\lambda^n
\]

since each \( q, \lambda \in R \). But this says that \( \lambda^{n+1} \in \lambda^n RbR = R(\lambda^n b)R = 0 \), a contradiction to \( \lambda \neq 0 \). Thus \( qJ = 0 \) and so \( q = 0 \).

The next theorem will be instrumental in proving conjecture II.

Theorem 3. Let \( J \) be any right (left) ideal of \( R \). Then either \( l(J) \neq 0 \) (\( r(J) \neq 0 \)) or \( J \) contains a two-sided essential ideal of \( R \).

Proof. By symmetry we may assume \( J \) is a right ideal. Suppose \( l(J) = 0 \). If \( A \) is an ideal of \( J \) such that \( A^2 = 0 \), then \( (AJ)^2 \subseteq A^2 = 0 \). Since \( R \) is semiprime we have \( AJ = 0 \), i.e., \( A \subseteq l(J) \). Therefore \( A = 0 \) and so \( J \) is a semiprime ring. Similarly \( r(J)J^2 = 0 \) and \( r(J)J = 0 \), forcing \( r(J) = 0 \).

Let \( U \) be the largest two-sided ideal of \( R \) which is contained in \( J \). Suppose \( U \) is not an essential ideal of \( R \). Then there is a nonzero ideal \( V \) such that \( UV = 0 \), \( V \cap J \neq 0 \), otherwise \( JV = 0 \), leading to \( VJ = 0 \), contradicting \( l(J) = 0 \). By Theorem A, applied to the semiprime ring \( J \) (and also using Lemma 1), \( V \cap J \cap C \neq 0 \). Picking \( 0 \neq \lambda \in V \cap J \cap C \), we obtain an ideal \( U + \lambda R \) of \( R \) lying in \( J \) and properly containing \( U \).

Theorem 1 and Theorem 3 together immediately imply.
Theorem 4. Let $J$ be an essential one-sided ideal of $R$. Then $J$ contains an essential two-sided ideal of $R$.

The following theorem answers conjecture II.

Theorem 5. $Q$ coincides with the maximal left quotient ring of $R$.

Proof. Utumi has shown [5, p. 145, Theorem 3.3] that if both singular ideals of $R$ are zero, then the maximal right and left quotient rings coincide if and only if for every nonessential right (left) ideal $J$ of $R$ the left (right) annihilator of $J$ is nonzero. But Theorem 3 precisely assures these conditions.

We recall from Lemma 1 that if $J$ is a right ideal of $R$ then $J \cap C$ is in fact the center of the ring $J$ and is an ideal of $C$.

Theorem 6. A right ideal $J$ is essential in $R$ if and only if $J \cap C$ is essential in $C$.

Proof. If $J$ is essential in $R$ then by Theorem 4 $J$ contains an essential 2-sided ideal $U$ of $R$. If $\lambda \neq 0 \in C$, $\lambda U$ is a nonzero ideal of $R$ since $U$ is essential. By Theorem A, $0 \neq \lambda u \in C$ for some $u \in U$. Suppose $\lambda (J \cap C) = 0$. Then $\lambda (U \cap C) = 0$, and in particular $(\lambda u)^2 = u\lambda (\lambda u) = 0$, a contradiction. Hence $\lambda (J \cap C) \neq 0$ and so $J \cap C$ is essential in $C$.

Conversely, if $J \cap C$ is essential in $C$, let $U$ be the ideal of $R$ generated by $J \cap C$. Suppose $r(U) \neq 0$. By Theorem A pick $0 \neq \lambda \in r(U) \cap C$. But then $(J \cap C) \lambda = 0$, a contradiction and so $r(U) = 0$. Thus $U$ and hence $J$ (which contains $U$) is essential in $R$.

Finally, we reprove some results due to Herstein and Small [3, p. 328, Theorems 2 and 3].

Theorem 7. If $l(a) = 0$, then $a$ is regular, $aR$ and $Ra$ are essential right and left ideals, and $a$ is invertible in $Q$.

Proof. By Theorem 3, $aR$ contains an essential ideal $U$ of $R$ and thus $aR$ is essential. Since $Q$ is (von Neumann) regular there exists $q \in Q$ such that $aq = a$. Thus $(aq - 1)aR = 0$ and so $aq = 1$, since $aR$ is essential. Suppose $qa \neq 1$. It is easily checked that $e_{ij} = q(1 - qa)a^j$, $i, j = 1, 2, \ldots$, is an infinite set of matrix units in $Q$. But this cannot occur, since $Q$ is a P.I. algebra by Theorem 2. Hence $qa = 1$, which implies that $a$ is regular and that $Ra$ is an essential left ideal of $R$.

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The referee pointed out that the proof of Theorem 2 can be adapted to yield the result that any polynomial identity satisfied by an essential right ideal of a semiprime P.I. ring $R$ must also be satisfied by $R$. He also indicated the connection between essential right ideals of $R$ and essential ideals of the center, and we carried out these suggestions in Theorem 6.
The proof of Theorem 3 was somewhat shortened following a suggestion by S. Steinberg. The latter also indicated that a more self-contained proof could be given for Theorem 5, without recourse to Utumi's paper but making use of Theorem 6. An error in our original proof of Theorem 7 was rectified by E. P. Armendariz. Finally we are indebted to the National Science Foundation, which partially supported the research involved in this paper (GP-12090).

REFERENCES


Department of Mathematics, University of Massachusetts, Amherst, Massachusetts 01002