PROBABILITY MEASURES ON SEMIGROUPS

PETER GERL

Abstract. Let \( S \) be a discrete semigroup, \( P \) a probability measure on \( S \) and \( s \in S \) with \( \lim \sup_{n \to \infty} \left( \frac{P^n(s)}{n} \right) = 1 \). We study limit theorems for the convolution powers \( P^n \) of \( P \) implied by the above property and further the class of all semigroups with this property. Theorem 3 relates this class of semigroups to left amenable semigroups.

1. Introduction. Let \( S \) be a discrete semigroup and \( P \) a probability measure on \( S \), that is a real valued function on \( S \) with \( P(s) \geq 0 \) for all \( s \in S \) and \( \sum_{s \in S} P(s) = 1 \). Kesten ([4], [5]) characterized amenable groups by means of the asymptotic behavior of convolution powers of symmetric probability measures defined on the group. A more precise information for the asymptotic behavior was obtained in [2] and [3] for symmetric probability measures on a discrete amenable group. In what follows we will derive similar theorems for probability measures on discrete semigroups.

Let \( S \) be a discrete semigroup, \( P \) a probability measure on \( S \). Then \( \text{Supp } P = \{ s \mid P(s) > 0 \} \) denotes the support of \( P \). To say \( \text{Supp } P \) generates \( S \) means: \( S = \bigcup_{n=1}^{\infty} \text{Supp } P^n \).

For probability measures \( P, Q \) on \( S \) define their convolution \( P \ast Q \) by

\[
P \ast Q(s) = \sum_{S_1 S_2 = s} P(s_1)Q(s_2)
\]

(the summation is to be extended over all representations of \( s \) as a product of two elements \( s_1, s_2 \) of \( S \)). \( P \ast Q \) is again a probability measure and \( \text{Supp } P \ast Q = (\text{Supp } P) \cdot (\text{Supp } Q) \). We often write \( P^{(1)} = P, P^{(n)} = P \ast P^{(n-1)} \).

Kesten obtained the following characterization of discrete amenable groups:

Let \( G \) be a discrete group with unit element \( e \), \( P \) a symmetric probability measure on \( G \) (\( P(g) = P(g^{-1}) \) for all \( g \) in \( G \)) such that \( G \) is generated by
Supp $P$; then

$$G \text{ is amenable} \iff P[e] = \limsup_{n \to \infty} (P^n(e))^{1/n} = 1.$$  

**2. Limit theorems.** Let $S$ be a discrete semigroup; if there exists a probability measure $P$ on $S$ with the properties

1. Supp $P$ generates $S$, and
2. there exists an $s \in S$ with $P[s] = \limsup_{n \to \infty} (P^n(s))^{1/n} = 1$ then we call $S$ an $A$-semigroup or, if we want to specify $P$, $(S, P)$ an $A$-pair.

Let $(S, P)$ be an $A$-pair with $P[s] = 1$ for some $s \in S$. Let $s' \in S$; since Supp $P$ generates $S$ there exists a natural number $k$ with $P^{(k)}(s') > 0$. Then

$$P^{(n+k)}(ss') = \sum_{s_1 \cdots s_k = ss'} P^n(s_1) P^{(k)}(s_2) \geq P^n(s) P^{(k)}(s')$$

and ($k$ is fixed)

$$1 \geq P[ss'] = \limsup_{n} (P^{(n+k)}(ss'))^{1/(n+k)} \geq \limsup_{n} (P^n(s))^{1/n} \lim(P^{(k)}(s'))^{1/n} = P[s] = 1.$$  

Therefore we have

**Proposition 1.** $P[s] = 1 \iff P[ss'] = P[s's] = P[s's's] = \ldots = 1$ for all $s', s'' \in S$.

$S$ is called left simple if for all $s \in S: Ss = S$ (this means every element of $S$ can be written in the form $s'$s). Proposition 1 implies

**Proposition 2.** (a) If $S$ is left simple (or right simple, or a group) then

$$P[s] = 1 \iff P[s] = 1 \quad \text{for every } s \in S.$$  

(b) If $S$ has a left unit $e$ ($es = s$ for all $s$) then

$$P[e] = 1 \iff P[s] = 1 \quad \text{for every } s \in S.$$  

Let $S$ be a discrete semigroup with a left unit $e$ and $(S, P)$ an $A$-pair, further put $P' = \frac{1}{2} (P + \delta_e)$ ($\delta_e$ is the probability measure concentrated at $e$, i.e. $\delta_e(e) = 1$, $\delta_e(s) = 0$ for $e \neq s \in S$). Then $P'$ is a probability measure on $S$ and Supp $P' = $ Supp $P \cup \{e\}$.

**Proposition 3.** $P[s] = 1 \Rightarrow P'[s] = 1$.

**Proof.** $e$ is a left unit, therefore $\delta_e \ast P = P$. So

$$P^{(2n)}(s) \geq \frac{1}{2^{2n}} \sum_{k=1}^{2n} \frac{1}{(2n)} \binom{2n}{k} P^{(k)}(s) \geq \frac{1}{4^n} \binom{2n}{n} P^n(s)$$

and

$$1 \geq (P^{(2n)}(s))^{1/2n} \geq \frac{1}{2} \left(\frac{1}{2^n} \binom{2n}{n} P^n(s)\right)^{1/2n} = a_n(P^n(s))^{1/2n}.$$  

Since $\lim_n a_n = 1$ we get $P'[s] = 1$.  

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Proposition 3 says that if \((S, P)\) is an \(A\)-pair then so is \((S, P')\) (and we have \(P'(e) > 0\)).

**Theorem 1.** Let \(S\) be a discrete semigroup with a left unit \(e\) and \((S, P)\) an \(A\)-pair. Then
1. \(P[e] = 1 \implies \lim_{n \to \infty} (P^{(n)}(s))^{1/n} = 1\) for every \(s \in S\).
2. \(\lim_{n \to \infty} (P^{(n)}(s))^{1/n} = 1 \iff \lim_{n \to \infty} (P^{(n+1)}(s)/P^{(n)}(s)) = 1\).

**Proof.** Similar to the proof of Theorem 1 and Theorem 2 of [3].

3. **The class of \(A\)-semigroups.**

**Theorem 2.** Let \(S\) be a finite semigroup and \(P\) a probability measure on \(S\) such that \(\text{Supp } P\) generates \(S\). Then \((S, P)\) is an \(A\)-pair.

**Proof.** Let \(c\) be the cardinal number of \(S\). Then for every \(n = 1, 2, \cdots\) there exists an element \(s_n\) in \(S\) with \(P^{(n)}(s_n) \geq 1/c\). Because \(S\) is finite there is an \(s_0\) in \(S\) which appears infinitely often in the sequence \(s_1, s_2, \cdots\) and so \(P^{(n)}(s_0) \geq 1/c\) for some sequence \(n_1 < n_2 < \cdots\) of natural numbers. Therefore \(P[s_0] = 1\).

**Theorem 3.** Let \(S\) be a discrete semigroup with left cancellation \((ss' = ss' \implies s' = s'')\) and a left unit \(e\). If \(S\) is an \(A\)-semigroup then \(S\) is left amenable.

**Proof.** By assumption there exists a probability measure \(P\) on \(S\) and an element \(s \in S\) such that (1) \(\text{Supp } P\) generates \(S\) and (2) \(P[s] = 1\). By Proposition 3 we have \(P'[s] = 1\) (\(P' = (P - \delta_e)\)).

Now let \(x \in l_2(S)\); then \(P' \ast x \in l_2(S)\) and \(\|P' \ast x\|_2 \leq \|P'\|_1 \|x\|_2 = \|x\|_2\). So we can consider \(P' \ast\) as an operator on \(l_2(S)\) and we have for its norm \(\|P' \ast\|_{2 \to 2} \leq 1\).

Further, \(\delta_e \in l_2(S)\). Next,
\[
P'(s) = P'(s)\delta_e(e) \leq \left(\sum_{s \in S} \left(\sum_{s_1, s_2, \cdots} P'(s_1)\delta_e(s_2)\right)^{1/2}\right) = \|P' \ast \delta_e\|_2,
\]
and in the same way
\[
P'(n)(s) \leq \|P'(n)\ast\|_{2 \to 2} \leq 1.
\]
So \(1 = P'[s] \leq \lim \sup_{n} \|P'(n) \ast\|_{2 \to 2}^{1/n} = \text{spectral radius of } P' \ast \leq \|P' \ast\|_{2 \to 2} \leq 1\) or \(\|P' \ast\|_{2 \to 2} = 1\); by the same argument \(\|P'^{(k)} \ast\|_{2 \to 2} = 1\) for \(k = 1, 2, \cdots\).

But \(\text{Supp } P'\) generates \(S\) and so for every finite \(E \subset S\) there exists a natural number \(K\) with \(E \subset \text{Supp } P'^{(k)}\) and \(e \in \text{Supp } P'^{(K)}\). Then [1] (Theorem 1, (e) \\(\Rightarrow\) (a)) implies that \(S\) is left amenable.

**Remark 1.** For \(S\) a group \(G\) and \(P\) a symmetric probability measure on \(G\) whose support generates \(G\) we have from the theorem of Kesten...
and Proposition 2: $G$ amenable $\Rightarrow P[g] = 1$ for every $g \in G$. So for groups we lose nothing in considering only symmetric probability measures. If $P$ is not symmetric this implication need no longer be true; consider for example the infinite cyclic group $G = \langle a \rangle$, generated by $a$. This group is commutative, therefore amenable. Let

$$P = \alpha \delta_a + (1 - \alpha)\delta_{a^{-1}} \quad (0 < \alpha < 1, \alpha \neq \frac{1}{2}).$$

Then

$$P^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \delta_a^{n-k} \alpha^k (1 - \alpha)^{n-k}$$

and

$$P^{(2n)}(e) = \binom{2n}{n} \alpha^n (1 - \alpha)^n.$$

So

$$P[e] = \lim_{n \to \infty} \left( \binom{2n}{n} \right)^{1/2n} (\alpha (1 - \alpha))^{1/2} = 2(\alpha (1 - \alpha))^{1/2} < 1 \quad \text{for } \alpha \neq \frac{1}{2}$$

(and $P[e] = P[g]$ for every $g \in G$ by Proposition 2).

Remark 2. The statement of Theorem 3 is false for arbitrary semigroups, for there are finite semigroups (which are $A$-semigroups by Theorem 2) that are not left (or right) amenable.

Remark 3. The converse of Theorem 3 is not true in general, for there are left amenable semigroups with left cancellation and a unit that are not $A$-semigroups.

Consider, for example, the infinite cyclic semigroup $S = \{e, a, a^2, \cdots \}$, generated by $e$ (unit) and $a$. $S$ is abelian and therefore amenable. Let $P$ be a probability measure on $S$ such that Supp $P$ generates $S$. This implies $0 < P(e) = \alpha < 1$ and so $P = \alpha \delta_e + (1 - \alpha)P_1$, where Supp $P_1 \subset \{a, a^2, \cdots \} = S - \{e\}$.

Then Supp $P^{(n)}_1 \subset \{a^n, a^{n+1}, \cdots \}$ and therefore

$$P^{(n)}_1 = \sum_{k=0}^{n} \binom{n}{k} P_1^{(k)} \alpha^{n-k} (1 - \alpha)^k.$$

This gives $P^{(n)}(e) = \alpha^n$ and $P[e] = \alpha < 1$.

For $l = 1, 2, \cdots$ we find for $n$ large enough

$$P^{(n)}(a^l) \leq \sum_{k=0}^{l} \binom{n}{k} P_1^{(k)}(a^l) \alpha^{n-k} (1 - \alpha)^k$$

$$\leq \sum_{k=0}^{l} \binom{n}{k} \alpha^{n-k} (1 - \alpha)^k \leq \alpha^n \binom{n}{l} \sum_{k=0}^{l} \left( \frac{1 - \alpha}{\alpha} \right)^k$$

and therefore $P[a^l] \leq \alpha < 1$. So $S$ is not an $A$-semigroup.
Theorem 4. The homomorphic image of an $A$-semigroup is an $A$-semigroup.

Proof. Let $(S, P)$ be an $A$-pair with $P[s]=1$ ($s \in S$). Let $\varphi : S \to S_1$ be a homomorphism onto the semigroup $S_1$. Define the probability measure $P_1$ on $S_1$ by

$$P_1(s_1) = P(\varphi^{-1}(s_1)) = \sum_{s \in \varphi^{-1}(s_1)} P(s).$$

Then by induction

$$P_1^{(n)}(s_1) = \sum_{s_1' s_1'' = s_1} \sum_{s'' \in \varphi^{-1}(s_1'')} P(s') \sum_{s'' \in \varphi^{-1}(s_1'')} P_1^{(n-1)}(s'') = \sum_{s'' \in \varphi^{-1}(s_1)} P(s') P_1^{(n-1)}(s'') = \sum_{s'' \in \varphi^{-1}(s_1)} P_1^{(n)}(s'),$$

and therefore $1 \geq P_1^{(n)}(s_1) \geq P^{(n)}(s)$ for $s \in \varphi^{-1}(s_1)$. Thus $P_1[s_1] = 1$ if $P[s]=1$ (where $\varphi(s)=s_1$).

Theorem 5. Let $(S_1, P_1)$ be an $A$-pair, $(S_2, P_2)$ be an $A$-pair such that for some $s_2 \in S_2: \lim_{n \to \infty} (P_2^{(n)}(s_2))^{1/n} = 1$. Then $(S_1 \times S_2, P_1 \times P_2)$ is an $A$-pair.

Proof. $\text{Supp } P_1 \times P_2$ generates $S_1 \times S_2$ and

$$1 \geq (P_1 \times P_2)[(s_1, s_2)] \geq P_1[s_1] \lim (P_2^{(n)}(s_2))^{1/n} = P_1[s_1] = 1$$

for some $s_1 \in S_1$.

Example 1. Let $S$ be a countable right zero semigroup ($ss'=s'$ for all $s, s' \in S$). If $P$ is any probability measure on $S$ whose support generates $S$, then $\text{Supp } P = S$ and

$$P^{(n)}(s) = \sum_{s_1 s_2 = s} P(s_1) P^{(n-1)}(s_2) = \sum_{s_1 \in S} P(s_1) P^{(n-1)}(s) = P(s).$$

Therefore $P[s] = \lim (P(s))^{1/n} = 1$, because $P(s) > 0$ for every $s \in S$; so we see that every countable right zero semigroup is an $A$-semigroup.

Example 2. As in Remark 3 one can show that the semigroup $S = \{e, a, b, ab, \cdots \}$, generated by two elements $a$ and $b$, is not an $A$-semigroup.

References


5. ———, *Full Banach mean values on countable groups*, Math. Scand. 7 (1959), 146–156. MR 22 #2911.

DEPARTMENT OF MATHEMATICS, PAHLAVI UNIVERSITY, SHIRAZ, IRAN

Current address: Mathematisches Institut der Universität, Strudlhofgasse 4, A-1090 Wien, Austria