ON COMPACT OPERATORS IN THE WEAK CLOSURE OF THE RANGE OF A DERIVATION

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Abstract. It is shown that if K is a compact operator which commutes with a bounded operator A on a Hilbert space H and if K is contained in the weak closure of the range of the derivation induced by A, then K is quasinilpotent.

Introduction and notations. Let H be a fixed separable infinite dimensional Hilbert space over the complex field C, and let \( \delta_A(X) = AX - XA \) be an inner derivation on the algebra \( \mathcal{L}(H) \) of all bounded operators on H. \( R(\delta_A)^\prime \) will denote the norm closure of the range of \( \delta_A \) in \( \mathcal{L}(H) \), and \( R(\delta_A)^\prime_{\text{w}} \) will denote the closure of \( R(\delta_A) \) in the weak operator topology. \( \{A\}' \) will denote the commutant of A.

Kleinecke (1957) and Shirokov (1956) independently proved I. Kaplansky's conjecture that \( R(\delta_A) \cap \{A\}' \) consists entirely of quasinilpotent operators.

In fact, using their result, it is rather simple to show that if \( AX_n - X_n A \) converges to B in \( \{A\}' \) in the norm topology for \( \|X_n\| \leq M, (M > 0) \), then B is again quasinilpotent, as Pearcy has noted in [6].

Recently, J. Anderson [1] proved that there exists an operator A in \( \mathcal{L}(H) \) such that \( I \in R(\delta_A)^\prime \). Therefore, an operator B in \( R(\delta_A)^\prime \cap \{A\}' \) is not necessarily quasinilpotent.

The purpose of this paper is to prove that if \{X_n\} is a sequence of operators such that \( AX_n - X_n A \) converges weakly to K in the weak operator topology where K is a compact operator in \( \{A\}' \), then K is quasinilpotent.

Lemma. Let \( A \in \mathcal{L}(H) \) and let \{X_n\} be a sequence of operators in \( \mathcal{L}(H) \) such that the sequence \( \{AX_n - X_n A\} \) converges weakly to an operator C in...
Then, \((A - \lambda)^jX_n - X_n(A - \lambda)^i\rightarrow j(A - \lambda)^{i-1}C\) in the weak operator topology for every positive integer \(j\) and every fixed scalar \(\lambda\). (It is assumed that \((A - \lambda)^0 = I\) if \(j = 0\).

**Proof by Induction.**  The case \(j = 1\) is obvious.

For \(j = 2\), multiply the commutator \(AX_n - X_nA = (A - \lambda)X_n - X_n(A - \lambda)\) on the left by \(A - \lambda\) and add it to the commutator multiplied on the right by \(A - \lambda\). Then, \((A - \lambda)^2X_n - X_n(A - \lambda)^2 \rightarrow 2(A - \lambda)C\) weakly. Suppose \((A - \lambda)^kX_n - X_n(A - \lambda)^k \rightarrow k(A - \lambda)^{k-1}C\) weakly for all \(k \leq j\) where \(j \geq 2\).

Then

\[
\begin{align*}
(A - \lambda)^{(A - \lambda)^jX_n - X_n(A - \lambda)^i} + \{(A - \lambda)^jX_n - X_n(A - \lambda)^i\}(A - \lambda) & = (A - \lambda)^{j+1}X_n - X_n(A - \lambda)^{j+1} \\
& + (A - \lambda)^{(A - \lambda)^jX_n - X_n(A - \lambda)^i}(A - \lambda)^{-1}\rightarrow (j+1)(A - \lambda)^jC weakly.
\end{align*}
\]

Since \((A - \lambda)^{j+1}X_n - X_n(A - \lambda)^{j+1} \rightarrow (j+1)(A - \lambda)^{j+2}C\) weakly by the induction assumption, \((A - \lambda)^{j+1}X_n - X_n(A - \lambda)^{j+1} \rightarrow (j+1)(A - \lambda)^jC\) weakly.

**Theorem 1.** Let \(A \in \mathcal{L}(H)\) and let \(\{X_n\}\) be a sequence of operators in \(\mathcal{L}(H)\) such that the sequence \(\{AX_n - X_nA\}\) converges weakly to an operator \(C\) of finite rank in \(\{A\}'\). Then \(C\) is nilpotent.

**Proof.** Let the range of \(C\) be \(R(C)\) and let \(\dim R(C) = k \geq 1\). If \(R(C^{i+1}) \neq R(C^i)\) for \(1 \leq i \leq k\), then clearly \(C\) is nilpotent. Hence, we assume the existence of the smallest positive integer \(q\) such that \(R(C^q) = R(C^{q+1})\). Then, \(R(C^q) = R(C^q) \neq \{0\}\) for all \(r \geq q\) and \(C\) restricted to \(R(C^q)\) is nonsingular. Also, \(A\) may be assumed to be nonsingular by replacing \(A\) by \(A - \lambda I\) if necessary. For notational convenience, we let \(R(C) = W\). Since \(C \in \{A\}'\), \(W\) is invariant under \(A\) and \(A(W) = W\).

Let \(\lambda_1, \lambda_2, \cdots, \lambda_p\) be the distinct eigenvalues of \(A|_W\) (i.e., \(A\) restricted to \(W\)) with respective algebraic multiplicities \(m_1, m_2, \cdots, m_p\). Then, by Jordan's theorem \(W\) is a direct sum (not necessarily orthogonal) of \(p\) subspaces \(M_1, M_2, \cdots, M_p\) with respective dimensions \(m_1, m_2, \cdots, m_p\) such that \(AM_i \subseteq M_i\) and \(A - \lambda_i\) is nilpotent on \(M_i\) with index \(j_i \leq m_i\) for \(i = 1, 2, \cdots, p\).

Since the index of \(A - \lambda_1\) on \(M_1\) is \(j_1 \geq 1\), there exists a vector \(y \in M_1\) such that \((A - \lambda_1)^{j_1-1}y \neq 0\) and \((A - \lambda_1)^{j_1}y = 0\). The fact that \(AX_n - X_nA \rightarrow C\) weakly implies that \(C^qAX_n - C^qX_nA = A(C^qX_n) - (C^qX_n)A \rightarrow C^{q+1}\) weakly. By the previous lemma,

\[
(A - \lambda_1)^{j_1}(C^qX_n) - (C^qX_n)(A - \lambda_1)^{j_1} \rightarrow j_1(A - \lambda_1)^{j_1-1}C^{q+1}
\] weakly.
Since \((C^qX_n)y \in W\), it is obvious that
\[(A - \lambda_1)^{j_1}(C^qX_n)y \in M_2 \oplus M_3 \oplus \cdots \oplus M_p.\]

Now \(j_1C^{q+1}(A-\lambda_1)^{j_1-1}y\) is a nonzero vector in \(M_1\), because of the way \(y\) was chosen and the nonsingularity of \(C^{q+1}|W\).

Since \(M_1, M_2, \cdots, M_p\) are fixed and \((A-\lambda_1)^{j_1}y = 0\),
\[
\inf_{n} \|(A - \lambda_1)^{j_1}(C^qX_n)y - j_1C^{q+1}(A - \lambda_1)^{j_1-1}y\| > 0.
\]

On the other hand, since \(W\) is finite dimensional and is invariant under \((A-\lambda_1)^{j_1}(C^qX_n), (C^qX_n)(A-\lambda_1)^{j_1}\), and \((A-\lambda_1)^{j_1-1}C^{q+1}\) for all \(n = 1, 2, \cdots\), \((A-\lambda_1)^{j_1}(C^qX_n) - (C^qX_n)(A-\lambda_1)^{j_1}|W\) converges strongly to
\[
j_1(A - \lambda_1)^{j_1-1}C^{q+1}|W.\]

Therefore,
\[
\lim_{n} \|(A - \lambda_1)^{j_1}(C^qX_n)y - j_1C^{q+1}(A - \lambda_1)^{j_1-1}y\| = 0.
\]

This is an obvious contradiction. Hence, all operators of finite rank in \(R(\delta_{A})^{\omega}\cap \{A\}'\) are nilpotent.

This result allows us to obtain the following stronger theorem.

**Theorem 2.** If \(A \in \mathcal{L}(H)\) and \(\{X_n\}\) is a sequence of operators in \(\mathcal{L}(H)\) such that \((AX_n - X_nA)\) converges weakly to a compact operator \(K\) in \(\{A\}'\), then \(K\) is quasinilpotent.

**Proof.** Suppose \(K\) is not quasinilpotent. Then, there exists an isolated nonzero eigenvalue \(\lambda_0\) in the spectrum \(\sigma(K)\) of \(K\). By a theorem of Riesz [7, p. 183], the root space \(M(\lambda_0)\) corresponding to \(\lambda_0\) of \(K\) (i.e., the subspace \(\{f \in H : (K-\lambda_0)^n f = 0 \text{ for some positive integer } n\}\) is finite dimensional, is invariant under \(K\), and has the property that \(K| M(\lambda_0)\) has singleton spectrum \(\{\lambda_0\}\). Since \(\sigma(K)\) is denumerable with 0 the only possible point of accumulation, it is possible to choose two disjoint simply connected open sets \(\mathcal{U}_1\) and \(\mathcal{U}_2\) such that \(\lambda_0 \in \mathcal{U}_1\) and \(\sigma(K) - \{\lambda_0\} \subset \mathcal{U}_2\). Let \(\gamma_1\) be a simple closed rectifiable curve lying in \(\mathcal{U}_1\) containing \(\lambda_0\) in its interior, and let \(\gamma_2\) be a simple closed rectifiable curve lying in \(\mathcal{U}_2\) containing \(\sigma(K) - \{\lambda_0\}\) in its interior.

By a theorem of Riesz [7, p. 421],

\[
E = \frac{1}{2\pi i} \int_{\gamma_1} (z - K)^{-1} \, dz
\]
is an idempotent which commutes with $K$, and the range of $E$ is $M(\lambda_0)$. Let $f(z)$ be the analytic function defined on $\mathcal{U}_1 \cup \mathcal{U}_2$ which is identically 1 on $\mathcal{U}_1$ and identically 0 on $\mathcal{U}_2$. It follows from the definition of $f(K)$ [7, p. 431] and (1) that $f(K) = E$. Since the complement of $\mathcal{U}_1 \cup \mathcal{U}_2$ is connected in the extended complex plane, it follows from Rung’s theorem [4, p. 317] that there exists a sequence of polynomials $P_n(z)$ which converges uniformly to $f(z)$ on $\gamma_1 \cup \gamma_2$.

Thus,

$$P_n(K) - E = P_n(K) - f(K)$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} (P_n(z) - f(z))(z - K)^{-1} dz$$

$$+ \frac{1}{2\pi i} \int_{\gamma_2} P_n(z)(z - K)^{-1} dz.$$

Since $\| (z - K)^{-1} \|$ is uniformly bounded on $\gamma_1 \cup \gamma_2$, it follows that $\| P_n(K) - E \| \to 0$. Now observe that the sequence $K(AX_n - X_nA) = A(KX_n) - (KX_n)A$ converges weakly to $K^2$. A similar argument shows that if $P$ is any polynomial, then $P(K)$ belongs to $R(\delta_A)^{-w} \cap \{A\}'$. In particular, if $P_n(z)$ is the sequence determined above, then $P_n(K) \in R(\delta_A)^{-w} \cap \{A\}'$. Therefore, $f(K) = E \in R(\delta_A)^{-w} \cap \{A\}'$. Since $E$ is an operator of finite rank and $1 \in \sigma(E)$, this contradicts Theorem 1. Therefore all compact operators $K$ in $R(\delta_A)^{-w} \cap \{A\}'$ are quasinilpotent.

Examination of the proof of Theorem 2 shows that the hypothesis that $K$ is compact was used only to conclude that the idempotent $E$ defined by (1) is an operator of finite rank. Thus, we have actually proved the following stronger result.

**Theorem 3.** Let $A \in \mathcal{L}(H)$ and let $T$ be any operator in $R(\delta_A)^{-w} \cap \{A\}'$. If $\lambda_0$ is any isolated point in $\sigma(T)$, and $\gamma$ is a simple closed rectifiable curve in the resolvent set of $T$ containing, in its interior, no other points of $\sigma(T)$ except $\lambda_0$, then the idempotent $E = (2\pi i)^{-1} \int_{\gamma} (z - T)^{-1} dz$ must have infinite dimensional range.

Some remarks. In [10] J. P. Williams proves that if $A$ and $B$ belong to $\mathcal{L}(H)$ and are not scalar multiples of the identity, then $R(\delta_A) \cap R(\delta_B) \neq 0$. Therefore, $R(\delta_A)$ is in general large and contains a subset of compact operators. But, in [9], Stampfli proved that no $R(\delta_A)$ is large enough to contain the ideal $F$ of all finite rank operators. The problem of classifying all noncompact operators in $R(\delta_A)^{-w} \cap \{A\}'$ for a given $A$ in $\mathcal{L}(H)$ is still an open problem.
BIBLIOGRAPHY


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