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ON COMPACT OPERATORS IN THE WEAK CLOSURE OF THE RANGE OF A DERIVATION

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ABSTRACT. It is shown that if K is a compact operator which commutes with a bounded operator A on a Hilbert space H and if K is contained in the weak closure of the range of the derivation induced by A, then K is quasinilpotent.

Introduction and notations. Let H be a fixed separable infinite dimensional Hilbert space over the complex field C, and let $\delta_A(X)=AX-XA$ be an inner derivation on the algebra $\mathscr{L}(H)$ of all bounded operators on H. $R(\delta_A)^-$ will denote the norm closure of the range of δ_A in $\mathscr{L}(H)$, and $R(\delta_A)^{-w}$ will denote the closure of $R(\delta_A)$ in the weak operator topology. $\{A\}'$ will denote the commutant of A.

Kleinecke (1957) and Shirokov (1956) independently proved I. Kaplansky's conjecture that $R(\delta_A)\cap\{A\}'$ consists entirely of quasinilpotent operators.

In fact, using their result, it is rather simple to show that if AX_n-X_nA converges to B in $\{A\}'$ in the norm topology for $\|X_n\|\leqq M$, $(M>0)$, then B is again quasinilpotent, as Pearcy has noted in [**6**].

Recently, J. Anderson [**1**] proved that there exists an operator A in $\mathscr{L}(H)$ such that $I\in R(\delta_A)^-$. Therefore, an operator B in $R(\delta_A)^-\cap\{A\}'$ is not necessarily quasinilpotent.

The purpose of this paper is to prove that if $\{X_n\}$ is a sequence of operators such that $(AX_n-X_nA)\to K$ in the weak operator topology where K is a compact operator in $\{A\}'$, then K is quasinilpotent.

LEMMA. *Let $A\in\mathscr{L}(H)$ and let $\{X_n\}$ be a sequence of operators in $\mathscr{L}(H)$ such that the sequence $\{AX_n-X_nA\}$ converges weakly to an operator C in*

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$\{A\}'$. Then, $(A-\lambda)^j X_n - X_n(A-\lambda)^j \to j(A-\lambda)^{j-1}C$ in the weak operator topology for every positive integer j and every fixed scalar λ. (It is assumed that $(A-\lambda)^j = I$ if $j=0$.)

PROOF BY INDUCTION. The case $j=1$ is obvious.

For $j=2$, multiply the commutator $AX_n - X_nA = (A-\lambda)X_n - X_n(A-\lambda)$ on the left by $A-\lambda$ and add it to the commutator multiplied on the right by $A-\lambda$. Then, $(A-\lambda)^2X_n - X_n(A-\lambda)^2 \to 2(A-\lambda)C$ weakly. Suppose $(A-\lambda)^kX_n - X_n(A-\lambda)^k \to k(A-\lambda)^{k-1}C$ weakly for all $k \leqq j$ where $j \geqq 2$.

Then

$$\begin{aligned}(A-\lambda)\{(A-\lambda)^jX_n - X_n(A-\lambda)^j\} + \{(A-\lambda)^jX_n - X_n(A-\lambda)^j\}(A-\lambda)\\ = (A-\lambda)^{j+1}X_n - X_n(A-\lambda)^{j+1}\\ + (A-\lambda)\{(A-\lambda)^{j-1}X_n - X_n(A-\lambda)^{j-1}\}(A-\lambda)\\ \to 2j(A-\lambda)^jC \text{ weakly.}\end{aligned}$$

Since $(A-\lambda)^{j-1}X_n - X_n(A-\lambda)^{j-1} \to (j-1)(A-\lambda)^{j-2}C$ weakly by the induction assumption, $(A-\lambda)^{j+1}X_n - X_n(A-\lambda)^{j+1} \to (j+1)(A-\lambda)^jC$ weakly.

THEOREM 1. *Let $A \in \mathscr{L}(H)$ and let $\{X_n\}$ be a sequence of operators in $\mathscr{L}(H)$ such that the sequence $\{AX_n - X_nA\}$ converges weakly to an operator C of finite rank in $\{A\}'$. Then C is nilpotent.*

PROOF. Let the range of C be $R(C)$ and let $\dim R(C) = k \geqq 1$. If $R(C^{i+1}) \neq R(C^i)$ for $1 \leqq i \leqq k$, then clearly C is nilpotent. Hence, we assume the existence of the smallest positive integer q such that $R(C^q) = R(C^{q+1})$. Then, $R(C^q) = R(C^r) \neq \{0\}$ for all $r \geqq q$ and C restricted to $R(C^q)$ is nonsingular. Also, A may be assumed to be nonsingular by replacing A by $A - \lambda I$ if necessary. For notational convenience, we let $R(C^q) = W$. Since $C \in \{A\}'$, W is invariant under A and $A(W) = W$.

Let $\lambda_1, \lambda_2, \cdots, \lambda_p$ be the distinct eigenvalues of $A|W$ (i.e., A restricted to W) with respective algebraic multiplicities $m_1, m_2, \cdots, m_p$. Then, by Jordan's theorem W is a direct sum (not necessarily orthogonal) of p subspaces $M_1, M_2, \cdots, M_p$ with respective dimensions $m_1, m_2, \cdots, m_p$ such that $AM_i \subseteq M_i$ and $A - \lambda_i$ is nilpotent on M_i with index $j_i \leqq m_i$ for $i = 1, 2, \cdots, p$.

Since the index of $A - \lambda_1$ on M_1 is $j_1 \geqq 1$, there exists a vector $y \in M_1$ such that $(A-\lambda_1)^{j_1-1}y \neq 0$ and $(A-\lambda_1)^{j_1}y = 0$. The fact that $AX_n - X_nA \to C$ weakly implies that $C^qAX_n - C^qX_nA = A(C^qX_n) - (C^qX_n)A \to C^{q+1}$ weakly. By the previous lemma,

$$(A-\lambda_1)^{j_1}(C^qX_n) - (C^qX_n)(A-\lambda_1)^{j_1} \to j_1(A-\lambda_1)^{j_1-1}C^{q+1}$$

weakly.

Since $(C^qX_n)y \in W$, it is obvious that

$$(A-\lambda_1)^{j_1}(C^qX_n)y \in M_2 \oplus M_3 \oplus \cdots \oplus M_p.$$

Now $j_1C^{q+1}(A-\lambda_1)^{j_1-1}y$ is a nonzero vector in M_1, because of the way y was chosen and the nonsingularity of $C^{q+1}|W$.

Since $M_1, M_2, \cdots, M_p$ are fixed and $(A-\lambda_1)^{j_1}y=0$,

$$\inf_n \|(A-\lambda_1)^{j_1}(C^qX_n)y - j_1C^{q+1}(A-\lambda_1)^{j_1-1}y\| > 0.$$

On the other hand, since W is finite dimensional and is invariant under $(A-\lambda_1)^{j_1}(C^qX_n)$, $(C^qX_n)(A-\lambda_1)^{j_1}$, and $(A-\lambda_1)^{j_1-1}C^{q+1}$ for all $n=1,2,\cdots$, $(A-\lambda_1)^{j_1}(C^qX_n)-(C^qX_n)(A-\lambda_1)^{j_1}|W$ converges strongly to

$$j_1(A-\lambda_1)^{j_1-1}C^{q+1} \mid W.$$

Therefore,

$$\lim_n \|(A-\lambda_1)^{j_1}(C^qX_n)y - j_1C^{q+1}(A-\lambda_1)^{j_1-1}y\| = 0.$$

This is an obvious contradiction. Hence, all operators of finite rank in $R(\delta_A)^{-w} \cap \{A\}'$ are nilpotent.

This result allows us to obtain the following stronger theorem.

THEOREM 2. *If $A \in \mathscr{L}(H)$ and $\{X_n\}$ is a sequence of operators in $\mathscr{L}(H)$ such that (AX_n-X_nA) converges weakly to a compact operator K in $\{A\}'$, then K is quasinilpotent.*

PROOF. Suppose K is not quasinilpotent. Then, there exists an isolated nonzero eigenvalue λ_0 in the spectrum $\sigma(K)$ of K. By a theorem of Riesz [7, p. 183], the root space $M(\lambda_0)$ corresponding to λ_0 of K (i.e., the subspace $\{f \in H: (K-\lambda_0)^n f=0$ for some positive integer $n\}$) is finite dimensional, is invariant under K, and has the property that $K \mid M(\lambda_0)$ has singleton spectrum $\{\lambda_0\}$. Since $\sigma(K)$ is denumerable with 0 the only possible point of accumulation, it is possible to choose two disjoint simply connected open sets $\mathscr{U}_1$ and $\mathscr{U}_2$ such that $\lambda_0 \in \mathscr{U}_1$ and $\sigma(K)-\{\lambda_0\} \subset \mathscr{U}_2$. Let γ_1 be a simple closed rectifiable curve lying in $\mathscr{U}_1$ containing λ_0 in its interior, and let γ_2 be a simple closed rectifiable curve lying in $\mathscr{U}_2$ containing $\sigma(K)-\{\lambda_0\}$ in its interior.

By a theorem of Riesz [7, p. 421],

$$E = \frac{1}{2\pi i}\int_{\gamma_1} (z-K)^{-1}\,dz \tag{1}$$

is an idempotent which commutes with K, and the range of E is $M(\lambda_0)$. Let $f(z)$ be the analytic function defined on $\mathscr{U}_1 \cup \mathscr{U}_2$ which is identically 1 on $\mathscr{U}_1$ and identically 0 on $\mathscr{U}_2$. It follows from the definition of $f(K)$ [7, p. 431] and (1) that $f(K)=E$. Since the complement of $\mathscr{U}_1 \cup \mathscr{U}_2$ is connected in the extended complex plane, it follows from Rung's theorem [**4**, p. 317] that there exists a sequence of polynomials $P_n(z)$ which converges uniformly to $f(z)$ on $\gamma_1 \cup \gamma_2$.

Thus,

$$\begin{aligned} P_n(K) - E &= P_n(K) - f(K) \\ &= \frac{1}{2\pi i} \int_{\gamma_1} (P_n(z) - f(z))(z - K)^{-1}\, dz \\ &\quad + \frac{1}{2\pi i} \int_{\gamma_2} P_n(z)(z - K)^{-1}\, dz. \end{aligned}$$

Since $\|(z-K)^{-1}\|$ is uniformly bounded on $\gamma_1 \cup \gamma_2$, it follows that $\|P_n(K)-E\| \to 0$. Now observe that the sequence $K(AX_n - X_nA) = A(KX_n) - (KX_n)A$ converges weakly to K^2. A similar argument shows that if P is any polynomial, then $P(K)$ belongs to $R(\delta_A)^{-w} \cap \{A\}'$. In particular, if $P_n(z)$ is the sequence determined above, then $P_n(K) \in R(\delta_A)^{-w} \cap \{A\}'$. Therefore, $f(K)=E \in R(\delta_A)^{-w} \cap \{A\}'$. Since E is an operator of finite rank and $1 \in \sigma(E)$, this contradicts Theorem 1. Therefore all compact operators K in $R(\delta_A)^{-w} \cap \{A\}'$ are quasinilpotent.

Examination of the proof of Theorem 2 shows that the hypothesis that K is compact was used only to conclude that the idempotent E defined by (1) is an operator of finite rank. Thus, we have actually proved the following stronger result.

THEOREM 3. *Let $A \in \mathscr{L}(H)$ and let T be any operator in $R(\delta_A)^{-w} \cap \{A\}'$. If λ_0 is any isolated point in $\sigma(T)$, and γ is a simple closed rectifiable curve in the resolvent set of T containing, in its interior, no other points of $\sigma(T)$ except λ_0, then the idempotent $E=(2\pi i)^{-1} \int_\gamma (z-T)^{-1}\, dz$ must have infinite dimensional range.*

Some remarks. In [**10**] J. P. Williams proves that if A and B belong to $\mathscr{L}(H)$ and are not scalar multiples of the identity, then $R(\delta_A) \cap R(\delta_B) \neq 0$. Therefore, $R(\delta_A)$ is in general large and contains a subset of compact operators. But, in [**9**], Stampfli proved that no $R(\delta_A)$ is large enough to contain the ideal F of all finite rank operators. The problem of classifying all noncompact operators in $R(\delta_A)^{-w} \cap \{A\}'$ for a given A in $\mathscr{L}(H)$ is still an open problem.

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