SOME PATHOLOGY INVOLVING PSEUDO /-GROUPS AS GROUPS OF DIVISIBILITY

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Abstract. In a partially ordered abelian group $G$, two elements $a$ and $b$ are pseudo-disjoint if $a, b \geq 0$ and either one is zero, or both are strictly positive and each $o$-ideal which is maximal with respect to not containing $a$ contains $b$, and vice versa. $G$ is a pseudo lattice-group if every element of $G$ can be written as a difference of pseudo-disjoint elements.

We prove the following theorem: suppose $G$ is an abelian pseudo lattice-group; if there is an $x > 0$ and a finite set of pairwise pseudo-disjoint elements $x_1, x_2, \ldots, x_k$ all of which exceed $x$, and in addition this set is maximal with respect to the above properties, then $G$ is not a group of divisibility.

The main consequence of this result is that every so-called "c-group" $V(\Lambda, R)$ for a given partially ordered set $\Lambda$, and where $R_\Lambda$ is a subgroup of the additive reals in their usual order, is a group of divisibility only if $\Lambda$ is a root system, and hence $V(\Lambda, R)$ is a lattice-ordered group. We do give examples of pseudo lattice-groups which are not lattice-groups, and yet are groups of divisibility.

Finally, we compute for each integral domain $D$ whose group of divisibility is a lattice-group, the group of divisibility of the polynomial ring $D[x]$ in one variable.

1. Preliminaries. All groups in this paper are abelian, and in additive notation unless otherwise indicated. An integral domain here shall be a commutative ring with identity and no zero divisors. If $D$ is an integral domain and $K$ is its quotient field, then the group of divisibility of $D$ is the multiplicative group of nonzero elements of $K$ modulo the group $U(D)$ of units of $D$; in symbols $G(D) \cong K^*/U(D)$. This group can be given a directed partial order by setting $xU(D) \leq yU(D)$ if $yx^{-1} \in D$. A (directed) p.o. group $G$ is called a group of divisibility if there is an integral domain $D$ such that $G \cong G(D)$. We can also view this concept in terms of semi-valuations: let $K$ be a field, $G$ be a directed p.o. group, and $v: K^* \to G$...
be a mapping onto $G$ satisfying
(i) $v(xy) = v(x) + v(y)$, for all $x, y \in K^*$;
(ii) $v(-1) = 0$;
(iii) $v(x+y) \geq g$ if $v(x), v(y) \geq g$, with $x, y \in K^*$ and $g \in G$.
Such a mapping is called a semivaluation. Let $D = \{x \in K^* | v(x) \geq 0\}$; then $D$ is a subring of $K$, $K$ is its quotient field and $G \cong G(D)$. Conversely, if $D$ is an integral domain and $K$ is its quotient field, then the canonical mapping $K^* \to G(D)$ is a semivaluation (see [5, p. 8]; also [9, p. 1148]). Consequently, $G$ is a group of divisibility if and only if there is a semi-
valuation onto $G$.

If $G$ is a totally ordered group (abbreviation $o$-group), the map $v$ is
called a valuation, and Krull [6, p. 164] demonstrated that every $o$-group
is a group of divisibility. Jaffard [4, p. 264] then showed that all lattice-
groups (abbreviation $l$-groups) are groups of divisibility.

In a p.o. group a directed, convex subgroup is called an $o$-ideal. Suppose
$G$ is a p.o. group and $0 \leq a, b \in G$; $a$ and $b$ are pseudo-disjoint if either is
zero, or both are strictly positive, and every $o$-ideal which is maximal
with respect to not containing $a$ contains $b$, and vice versa. A pseudo
lattice-group (abbreviation pseudo $l$-group) is a p.o. group in which every
element can be written as the difference of two pseudo-disjoint elements.
For the basic material concerning pseudo $l$-groups we refer the reader to
[1] and [3]. Conrad shows in [1] that in a pseudo $l$-group $G$, $0 \leq a, b \in G$
are pseudo-disjoint if and only if $c \leq a, b$ implies that $nc \leq a, b$ for each
positive integer $n$.

For a given partially ordered set $\Lambda$, and each $\lambda \in \Lambda$, let $R_\lambda$ be a subgroup
of the additive real numbers equipped with the usual order. Form $V(\Lambda, R_\lambda)$: the subgroup of the cartesian product of the $R_\lambda$ over $\Lambda$ consisting of the
"vectors" $v = (\cdots, v_1, \cdots)$ whose supports have no infinite ascending
chains. $V(\Lambda, R_\lambda)$ becomes a p.o. group by setting $0 < v = (\cdots, v_1, \cdots)$
if $v_\lambda > 0$ for each maximal component $\lambda$ of the support of $v$. Then $V(\Lambda, R_\lambda)$ is a pseudo $l$-group (see Theorem 4.8 in [1]), and every pseudo $l$-group
may be embedded in some $V(\Lambda, R_\lambda)$ so as to preserve pseudo-disjointness
(see 4.11 in [1]). It is well known that $V(\Lambda, R_\lambda)$ is an $l$-group if and only
if $\Lambda$ is a root system: $\{\lambda \in \Lambda | \lambda \geq \lambda_0\}$ is a chain for each $\lambda_0 \in \Lambda$. Finally, two elements $0 < v, w \in V(\Lambda, R_\lambda)$ are pseudo-disjoint if and only if no
maximal component of the support of $v$ is comparable to one in the
support of $w$ [1, p. 214].

2. The main theorem. We state our main result at the outset.

Theorem A. Suppose $G$ is a pseudo $l$-group, and there is an element
$0 < x \in G$ and a set $x_1, x_2, \cdots, x_k$ of pairwise pseudo-disjoint elements
all of which exceed $x$, and suppose further that this set is maximal with
respect to the above properties. Then $G$ is not a group of divisibility.
The proof depends on two lemmas, one rather interesting in its own right, the other rather technical.

**Lemma 1.** Suppose $G$ is a pseudo $l$-group, and $v$ is a semivaluation from a field $K$ upon $G$. If $0 < a, b \in G$ are pseudo-disjoint and $0 < c < a, b$, then there is an element $0 < g \in G$, pseudo-disjoint to $a$ and $b$, with $c < g$.

**Proof.** Let $v(x) = a$, $v(y) = b$ and $g = v(x + y)$. If $c \leq a$, then $c \leq v(-x)$, so that $b = v(y) = v(x + y - x) \leq c$. But $a$ and $b$ are pseudo-disjoint and hence $nc \leq a, b$, for any positive integer $n$. Again using one of the defining properties of semivaluations $nc \leq g$. Conclusion: $a$ and $g$ are pseudo-disjoint; likewise $b$ and $g$ are pseudo-disjoint. It is clear that if $c < a, b$ then $c < g$; in particular $g > 0$.

If $G$ is a pseudo $l$-group and $0 \neq x \in G$ we call an $o$-ideal $M$ of $G$ which is maximal with respect to not containing $x$ a value of $x$. In this language then, $a$ is pseudo-disjoint to $b$ if and only if every value of $a$ contains $b$, and vice versa.

**Lemma 2.** Suppose $G$ is a pseudo $l$-group and $0 < a \in G$, $0 < b_{i} \in G$ $(i=1, \ldots, k)$. Assume further that the $b_{i}$ are pairwise pseudo-disjoint, while $a$ is pseudo-disjoint to $b_{1} + b_{2} + \cdots + b_{k}$. Then $a$ is pseudo-disjoint to each $b_{i}$.

**Proof.** Let $M$ be a value of $a$; then by our assumption $b_{1} + b_{2} + \cdots + b_{k}$ is in $M$, and so by convexity each $b_{i} \in M$. On the other hand if $N$ is a value of $b_{i}$, each $b_{j} \in N$, for $j \neq i$; this makes $N$ a value of $b_{1} + b_{2} + \cdots + b_{k}$, and hence $a \in N$. It follows then that each $b_{i}$ is pseudo-disjoint to $a$.

**Proof of Theorem A.** Suppose $G$ is a pseudo $l$-group, $0 < x \in G$ and $x_{1}, x_{2}, \ldots, x_{k}$ is a maximal, pairwise pseudo-disjoint set of elements of $G$ exceeding $x$. Relabel $x_{1} = a$ and $b = x_{2} + x_{3} + \cdots + x_{k}$; then $a$ and $b$ are pseudo-disjoint.

If $G$ is a group of divisibility as well, there is semivaluation $v$ from a field $K$ onto $G$. By Lemma 1 we may find $0 < g \in G$ pseudo-disjoint to both $a$ and $b$, such that $x < g$. By Lemma 2 $g$ is pseudo-disjoint to each $x_{i}$ $(i=1, \ldots, k)$; this contradicts the maximality of the set $x_{1}, x_{2}, \ldots, x_{k}$ over $x$.

This proves the theorem.

Our first corollary concerns $v$-groups.

**Theorem B.** Let $\Lambda$ be a partially ordered set, $R_{\lambda}$ be an ordered subgroup of the reals for each $\lambda \in \Lambda$; set $V = V(\Lambda, R_{\lambda})$. If $V$ is a group of divisibility then $\Lambda$ is a root system and hence $V$ is an $l$-group.
Proof. If \( \Lambda \) is not a root system there exists a \( \nu \in \Lambda \) with pairwise incomparable elements above \( \nu \) in \( \Lambda \). Let \( \{ \lambda_i | i \in I \} \) be a set of mutually incomparable elements of \( \Lambda \) all of which exceed \( \nu \), and suppose \( \{ \lambda_i | i \in I \} \) is also maximal with respect to these properties. Fix \( j \in I \) and define \( a, b \in \nu \) as follows:

\[
\begin{align*}
a_\lambda &= 1, & \text{if } \lambda = \lambda_j, \\
b_\lambda &= 1, & \text{if } \lambda = \lambda_i, \; i \neq j, \\
&= 0, & \text{otherwise}.
\end{align*}
\]

Clearly \( 0 < a, b \in \nu \) and \( a \) is pseudo-disjoint to \( b \); moreover the pair \( \{a, b\} \) satisfies the conditions of Theorem A relative to, say, \( x \in \nu \), where

\[
\begin{align*}
x_\lambda &= 1, & \text{if } \lambda = \nu, \\
&= 0, & \text{otherwise}.
\end{align*}
\]

By the theorem we obtain a contradiction: for if there is an element \( 0 < g \in G \), pseudo-disjoint to both \( a \) and \( b \) which exceeds \( x \), then we contradict the maximality of the set \( \{ \lambda_i | i \in I \} \) over \( \nu \). Thus \( \nu \) cannot be a group of divisibility unless \( \Lambda \) is a root system.

If \( G \) is a pseudo \( l \)-group and \( 0 < u \in G \) has the property that no strictly positive element is pseudo-disjoint to \( u \), we call \( u \) a weak order unit.

Corollary 1. Suppose the pseudo \( l \)-group \( G \) has a weak order unit \( u \) which can be written as the sum of a pair of pseudo-disjoint elements which are not disjoint. Then \( G \) is not a group of divisibility.

Proof. Write \( u = a + b \) with \( a, b > 0 \) in \( G \) as prescribed in the statement of the corollary, and suppose \( 0 < c < a, b \). Then \( \{a, b\} \) is a maximal pseudo-disjoint set over \( c \), and Theorem A applies.

Let \( G \) be a p.o. group and \( A \) be an \( o \)-ideal of \( G \). We call \( G \) a lex-extension of \( A \) (by \( G/A \)) if for each \( 0 < a \in A \) and \( 0 < g \in G \\backslash A \), \( g > a \). \( G \) is a direct lex-extension of \( A \) if \( A \) is a direct summand: equivalently, \( G = B \oplus A \) and \( 0 \leq g = (b, a) \) if and only if \( b > 0 \), or \( b = 0 \) and \( a \geq 0 \). We then write \( G = B \bar{x} A \). If \( A \) and \( B \) are \( l \)-groups then \( G = B \bar{x} A \) is a pseudo \( l \)-group [3], and under these assumptions \( G \) is an \( l \)-group if and only if \( A = 0 \) or \( B \) is an \( o \)-group.

Call a weak order unit \( u \) in an \( l \)-group \( B \) decomposable if \( u \) can be written as a sum of pairwise disjoint, strictly positive elements of \( B \).

Corollary 2. Let \( A \neq 0 \) and \( B \) be \( l \)-groups, and suppose that \( B \) has a decomposable weak unit. Then \( G = B \bar{x} A \) is not a group of divisibility.

We compare our last corollary with Ohm's theorem 5.3 in [8]. Consider his condition labeled (5.1): there exist \( b_1, b_2 \in B \) such that \( b_1 \) and \( b_2 \) are

\[\text{1 We may assume without loss of generality that the number 1 is in each } R_A.\]
incomparable, and a subdirect representation of $B$ as a subdirect product of $o$-groups $B_i$ ($i \in I$) by an $l$-isomorphism $\sigma$ such that $b_1\sigma_i \not= b_2\sigma_i$, for all $i \in I$. It is equivalent to the existence of a decomposable weak order unit in $B$.

To see this note that if Ohm's (5.1) holds for an $l$-group $B$, and $b_1$ and $b_2$ are as specified above, then if we set $u = (b_1 - b_2)v_0 + (b_2 - b_1)v_0$, $u$ is a decomposable weak order unit. For $u\sigma_i = (b_1 - b_2)\sigma_i v_0 + (b_2 - b_1)\sigma_i v_0$, and so $u\sigma_i = (b_1 - b_2)\sigma_i$ or $(b_2 - b_1)\sigma_i$, either of which is $> 0$. Hence $u$ is a weak order unit, and it is clearly decomposable.

Conversely, suppose $B$ has a decomposable weak order unit $u$, and $u = a + b$, with $0 < a, b \in B$ and $a \land b = 0$. If a minimal prime subgroup $N$ of $B$ contains $u$ then by the minimality of $N$ there exists an element $0 < x \in B \setminus N$ such that $x \land u = 0$, a contradiction. Consider then the family $\{N_\lambda | \lambda \in \Lambda\}$ of minimal prime subgroups of $B$; let $B_\lambda = B/N_\lambda$ and $\sigma : B \to \prod B_\lambda$ be the induced $l$-embedding. Each $B_\lambda$ is an $o$-group and $u\sigma_\lambda > 0$, for each $\lambda \in \Lambda$. Let $b_1 = a - b$ and $b_2 = 0$; then this pair satisfies Ohm's condition relative to the mapping $\sigma$. (We refer the reader to [2, pp. 1.14-1.15 and pp. 2.13-2.14].)

His Theorem 5.3 is somewhat more general than Corollary 2 in view of the fact that we assume $A$ to be an $l$-group, whereas he does not.

Following Corollary 3.3 in [8] Ohm remarks that if one takes the polynomial ring $k[x, y]$ in two indeterminates over the field $k$, and localizes by the ideal generated by $x$ and $y$, one obtains a local ring whose group of divisibility is a cardinal sum of copies of $Z$, the integers in their usual order; the number of copies of $Z$ is at least 2 since the local ring is not a valuation ring. If $G$ is then the group of divisibility of a domain $D$ whose quotient field is $k$, Corollary 3.3 in [8] shows that the direct lex-extension of $G$ by this cardinal sum of integers is again a group of divisibility. If $G$ is an $l$-group such a lex-extension is a pseudo $l$-group which is not an $l$-group, providing a large class of examples of such pseudo $l$-groups which are groups of divisibility. In view of the observation in §1 that every pseudo $l$-group can be embedded in a reasonably “nice” way in a $v$-group, the examples here contrasted with Theorem B leave a rather monstrous question mark as to the nature of groups of divisibility, not only in the context of pseudo $l$-groups, but in general as well.

3. Polynomial rings and Gauss' lemma. We conclude this note with a result that calculates for an integral domain $D$ whose group of divisibility is an $l$-group, the group of divisibility of its polynomial ring $D[x]$ in one variable. Curiously, an analogue of the classical Gauss lemma for

\[^2\text{In view of Theorem A there are infinitely many copies of } Z \text{ in these cardinal sums.}\]
polynomials crops up at a rather crucial juncture. First, a general preliminary remark:

**Proposition.** Let \( D \) be an integral domain, \( G \) be its group of divisibility; then \( G(D[x]) \) is a direct extension of \( G \) by a cardinal sum of copies of \( Z \).

**Proof.** Let \( k \) be the quotient field of \( D \). We note here that the group of units \( U(D) \) of \( D \) is also the group of units of \( D[x] \). Further \( D[x] \) and \( k[x] \) have same quotient field, namely \( k(x) \), the field of rational functions in \( x \) with coefficients in \( k \). Finally, the group of units of \( k[x] \) is \( k^* \). Thus

\[
G = k^*/U(D), \quad G(D[x]) = (k(x))^*/U(D), \quad \text{and} \quad G(k[x]) = k(x)^*/k^*,
\]

and the latter is a cardinal sum of integers; see [7, Theorem 4.3]. Clearly, the inclusion of \( G \) in \( G(D[x]) \) is a convex order embedding, and the canonical epimorphism \( G(D[x]) \to G(k[x]) \) is an \( \sigma \)-epimorphism. Hence \( G(D[x])/G \cong G(k[x]) \); since \( G(k[x]) \) is abstractly a free abelian group, the extension is direct.

Now suppose \( G = G(D) \) is an \( \ell \)-group; then \( D \) has the following properties:

1. any finite set of nonzero elements of \( D \) has a greatest common divisor, and
2. if \( d \) divides \( ab \) (\( a, b, d \in D \)) then \( d = xy \) where \( x \) divides \( a \) and \( y \) divides \( b \). This is so because \( G \), being an \( \ell \)-group, satisfies the Riesz interpolation property: if \( 0 \leq a_1, a_2 \in G \) and \( 0 \leq b \in G \), then \( b \leq a_1 + a_2 \) implies that \( b = b_1 + b_2 \), with \( 0 \leq b_1 \leq a_i \) (\( i = 1, 2 \)).

Call a polynomial \( p(x) \) in \( D[x] \) primitive if the greatest common divisor of the coefficients of \( p(x) \) is a unit of \( D \). If \( G \) is an \( \ell \)-group any polynomial \( g(x) \in D[x] \) can be written uniquely (up to units) as \( g(x) = d \cdot g_0(x) \), where \( g_0(x) \) is primitive and \( d \) is the greatest common divisor of the coefficients of \( g(x) \).

The following is a crucial lemma.

**Lemma 3 (Gauss' Lemma).** If the group of divisibility \( G \) of an integral domain \( D \) satisfies the Riesz interpolation property, the product of two primitive polynomials in \( D[x] \) is primitive.

**Proof.** Let \( p(x) = a_0 + a_1 x + \cdots + a_m x^m \) and \( q(x) = b_0 + b_1 x + \cdots + b_n x^n \) be primitive polynomials, and \( p(x)q(x) = c_0 + c_1 x + \cdots + c_{m+n} x^{m+n} \). Suppose \( d \in D \) divides all \( c_k \), and is not a unit. Let \( i_0 \) (\( j_0 \)) be the first index such that \( d \) fails to divide \( a_{i_0} (b_{j_0}) \); set \( k_0 = i_0 + j_0 \). Then \( d \) divides \( c_{k_0} = a_{i_0} b_{j_0} + \cdots + a_{i_0} b_{j_0} + \cdots + a_{k_0} b_{j_0} \), and so \( d \) divides \( a_{i_0} b_{j_0} \). Since \( G \) satisfies the Riesz interpolation property \( d = x_0 y_0 \) where \( x_0 \) (\( y_0 \)) divides \( a_{i_0} (b_{j_0}) \). Now \( x_0 \) divides each \( c_k \), each \( a_i \) for \( i = 0, 1, \ldots, i_0 \) and each \( b_j \) for \( j = 0, 1, \ldots, j_0 - 1 \).
By induction, $x_0$ is a unit and so $d$ divides $b_{i_0}$, which is a contradiction. We conclude that $p(x)q(x)$ is primitive, and the lemma is proved.

**Theorem C.** If the group of divisibility $G$ of the integral domain $D$ is an $l$-group, then $G(D[x])$ is a cardinal sum of $G$ with a cardinal sum of copies of $Z$; in particular $G(D[x])$ is an $l$-group.

**Proof.** Recall that a saturated multiplicative system of an integral domain is a subset of nonzero elements, closed under multiplication, which contains along with an element $d$ all the divisors of $d$. Mott (see [7, Theorem 5.1]) showed that there is a natural isomorphism between the lattice of saturated multiplicative systems of an integral domain and the $o$-ideals of its group of divisibility.

Lemma 3 says that the subset $S$ of primitive polynomials in $D[x]$ is multiplicative; it is clearly saturated. Also, the nonzero elements of $D$ form a multiplicative system in $D[x]$ which is saturated; denote this subset by $D^*$. Since $G$ is an $l$-group we may write every nonzero polynomial $f(x)$ as a product of an element from $D^*$ and an element of $S$; evidently $S \cap D^* = U(D)$. By Mott's Theorem (and the logical extension thereof) there exist $o$-ideals $A$ and $B$ of $G(D[x])$ such that $G(D[x])$ is the cardinal sum of $A$ and $B$; if $A$ corresponds to $D^*$ then clearly $A \cong G$, and it is immediate that $B$ (corresponding to $S$) is isomorphic to $G(k[x])$. This concludes the proof of Theorem C.

We offer the following remark in the way of a converse of Theorem C. Let $D$ be an integral domain and $G$ be its group of divisibility. Without any further assumptions $G$ is an $o$-ideal of $G(D[x])$; so suppose it splits off cardinally. Then $G(D[x]) = G \uplus M$, where $M$ is an $o$-ideal of $G(D[x])$; using Mott's correspondence again we come up with a saturated multiplicative system $T$ in $D[x]$ having the properties that (1) $D^* \cap T = U(D)$ and (2) every nonzero polynomial $f(x)$ can be written (uniquely up to units) as the product of an element of $D^*$ and one from $T$. Now let $S$ be the set of primitive polynomials; clearly $S \subseteq T$, and if $p(x) \in T$ but is not primitive, then write $p(x) = d \cdot q(x)$, and pick $d$ to be a nonunit of $D$. Since $T$ is saturated $q(x) \in T$, but this violates the uniqueness of such expressions. Hence $T = S$.

Moreover pick $0 \neq a, b \in D$ and consider $f(x) = a + bx$; by writing $f(x)$ as a product of an element from $D^*$ and an element from $S$ we locate the greatest common divisor of $a$ and $b$. We can therefore make the following conclusion.

**Theorem D.** Let $G$ be the group of divisibility of the integral domain $D$; let $H = G(D[x])$. If $H$ is the cardinal sum of $G$ and $G(k[x])$ then

1. any finite set of nonzero elements of $D$ has a greatest common
divisor, and

(2) the subset \( S \) of primitive polynomials over \( D \) is a saturated multiplicative system.

If \( G \) satisfies the Riesz interpolation property it is an \( l \)-group.

Finally, in view of Theorem C conditions (1) and (2) are sufficient to insure that \( G \) split as a cardinal summand of \( H \).

In closing we pose one of many questions that arise naturally here: if \( G = G(D) \) satisfies the Riesz interpolation property, then does \( G(D[x]) \)?

Bibliography

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