IDENTITIES FOR SERIES OF THE TYPE \( \sum f(n)\mu(n)n^{-s} \)

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Abstract. Identities are obtained relating the series of the title with \( \sum f(n)\mu(n)\mu(p, n)n^{-s} \) where \( f \) is completely multiplicative, \( |f(n)| \leq 1 \), and \( p \) is prime. Applications are given to vanishing subseries of \( \sum \mu(n)/n \).


\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0,
\]

where \( \mu(n) \) is the Möbius function. Landau [5] later showed that (1) is equivalent to the prime number theorem. Kluyver [2] described a method for evaluating subseries of (1) of the form

\[
\sum_{m=0}^{\infty} \frac{\mu(mb + h)}{mb + h},
\]

where \( 0 < h \leq b \), although he did not prove convergence of these subseries. Landau [4] proved convergence and expressed the subseries (2) as a linear combination of reciprocals of Dirichlet \( L \)-functions.

The results of Landau and Kluyver imply the formulas

\[
\sum_{n=1; n=0 \mod p}^{\infty} \frac{\mu(n)}{n} = 0, \quad \sum_{n=1; n \neq 0 \mod p}^{\infty} \frac{\mu(n)}{n} = 0,
\]

for every prime \( p \). In this note we obtain some identities for Dirichlet series, one of which gives a new proof of (3).

**Theorem 1.** For any prime \( p \) and any complex \( s=\sigma+it \) with \( \sigma \geq 1 \) we have

\[
(1 + p^{-s}) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = (1 - p^{-s}) \sum_{n=1}^{\infty} \frac{\mu(n)\mu(p, n)}{n^s},
\]

where \( \mu(p, n) \) denotes the Möbius function evaluated at the g.c.d. of \( p \) and \( n \).

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Since
\[ \mu(p, n) = \begin{cases} 1 & \text{if } p \nmid n, \\ -1 & \text{if } p \mid n, \end{cases} \]
the relations in (3) follow by taking \( s = 1 \) in (4) and using (1).

A special case of Theorem 1 with \( p = 2 \) was recently discovered by Tord Hall [1]. Although Hall's method can be adapted to prove Theorem 1, the proof given here seems more natural. It is based on the following property of the Möbius function.

**Lemma 1.** For every prime \( p \) we have
\[ \sum_{d \mid n} \mu(d) \mu(p, d) = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = p^a, \ a \geq 1, \\ 0 & \text{otherwise}. \end{cases} \]

2. **Proof of Lemma 1.** If \( n = 1 \) the proof is immediate. If \( n > 1 \) we have
\[ \sum_{d \mid n} \mu(d) \mu(p, d) = \sum_{d \mid n : p \nmid d} \mu(d) - \sum_{d \mid n : p \mid d} \mu(d) = \sum_{d \mid n} \mu(d) - 2 \sum_{d \mid n : p \mid d} \mu(d) = -2 \sum_{d \mid n : p \mid d} \mu(d), \]
since \( n > 1 \). If \( p \nmid n \) the last sum is empty and hence equals zero. If \( p \mid n \) then \( n = p^a q \) where \( a \geq 1, \ (q, p) = 1, \ q > 1 \). Every divisor of \( n \) divisible by \( p \) has the form \( p^t \delta \) where \( 1 \leq t \leq a \) and \( \delta \mid q \). Hence the last sum is
\[ -2 \sum_{t=1}^{a} \sum_{\delta \mid q} \mu(p^t \delta) = -2 \sum_{\delta \mid q} \mu(p \delta) = 2 \sum_{\delta \mid q} \mu(\delta) = 2 \quad \text{if } q = 1, \]
\[ = 0 \quad \text{if } q > 1. \]

This proves Lemma 1.

3. **Proof of Theorem 1.** The sum in Lemma 1 is the coefficient of \( n^{-s} \) in the Dirichlet series obtained by multiplying \( \sum \mu(n) \mu(p, n)n^{-s} \) by \( \zeta(s) = \sum n^{-s} \). Therefore if \( s > 1 \) we have
\[ \zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n) \mu(p, n)}{n^s} = 1 + 2 \sum_{a=1}^{\infty} \frac{1}{p^{as}}. \]

The Dirichlet series on the right of (5) is also a geometric series which converges absolutely for \( s > 0 \) and has sum
\[ 1 + 2p^{-s}/(1 - p^{-s}) = (1 + p^{-s})/(1 - p^{-s}). \]
Since \( \frac{1}{\zeta(s)} = \sum \mu(n)n^{-s} \), equation (5) is equivalent to (4) for \( \sigma > 1 \). Now the series \( \sum \mu(n)\mu(p,n)n^{-s} \) also converges for \( \sigma = 1 \) since it is the product of the Dirichlet series \( \sum \mu(n)n^{-s} \), convergent for \( \sigma \geq 1 \), and a Dirichlet series which converges absolutely for \( \sigma > 0 \). (See Landau [6, §185].) Therefore the identity in (4) is valid for \( \sigma \geq 1 \).

Theorem 1 can be extended as follows.

**Theorem 2.** Let \( f \) be a completely multiplicative function with \( |f(n)| \leq 1 \) for all \( n \geq 1 \). Then for any prime \( p \) and any complex \( s=\sigma+it \) with \( \sigma > 1 \) we have

\[
(1 + f(p)p^{-s}) \sum_{n=1}^{\infty} \frac{f(n)\mu(n)}{n^s} = (1 - f(p)p^{-s}) \sum_{n=1}^{\infty} \frac{f(n)\mu(n)\mu(p,n)}{n^s}.
\]

Moreover, if the series \( \sum f(n)\mu(n)n^{-s} \) converges for \( \sigma \geq c \) for some \( c \) with \( 0 < c \leq 1 \), then (6) also holds for \( \sigma \geq c \).

**Proof.** If \( f(n) = 0 \) for all \( n \) the result holds trivially. If not, then \( f(1) = 1 \) and by Lemma 1 we have

\[
\sum_{d|n} f(d)\mu(d)\mu(p,d)f(n/d) = f(n) \sum_{d|n} \mu(d)\mu(p,d) = 1 \quad \text{if } n = 1,
\]

\[
= 2f(p)^a \quad \text{if } n = p^a, \quad a \geq 1,
\]

\[
= 0 \quad \text{otherwise}.
\]

This identity implies, for \( \sigma > 1 \),

\[
\left( \sum_{n=1}^{\infty} \frac{f(n)\mu(n)\mu(p,n)}{n^s} \right) \left( \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \right) = 1 + 2 \sum_{a=1}^{\infty} \frac{f(p)^a}{p^{as}}.
\]

Since \( |f(n)| \leq 1 \), each Dirichlet series on the left converges absolutely for \( \sigma > 1 \), and the geometric series on the right converges absolutely for \( \sigma > 0 \) to the sum \( (1+f(p)p^{-s})/(1-f(p)p^{-s}) \). Also, \( \sum_{n=1}^{\infty} f(n)n^{-s} \neq 0 \) for \( \sigma > 1 \) since it has an Euler product, and

\[
\left( \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \right)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)f(n)}{n^s}.
\]

The rest of the proof is like that of Theorem 1.

4. **Related results.** Sats 2 in Tord Hall’s paper is the special case \( p=2 \) of the following identity.

**Theorem 3.** For any prime \( p \) and \( s=\sigma+it \) with \( \sigma > 1 \), we have

\[
(1 - p^{-s}) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = (1 + p^{-s}) \sum_{n=1}^{\infty} \frac{\mu(n)\mu(p,n)}{n^s}.
\]
This theorem can be proved by Hall's method or by use of the following arithmetical identity.

**Lemma 2.** For all $n \geq 1$ and any prime $p$ we have

$$
\sum_{d|n} a(d) |\mu(n/d)| = \sum_{d|n} b(d) |\mu(n/d)| \mu(p, n/d),
$$

where

$$
a(n) = \begin{cases} 1 & \text{if } n = 1, \\ -1 & \text{if } n = p, \\ 0 & \text{otherwise,} \end{cases} \quad b(n) = \begin{cases} 1 & \text{if } n = 1, \\ 1 & \text{if } n = p, \\ 0 & \text{otherwise.} \end{cases}
$$

It is clear that Lemma 2 implies Theorem 3. To prove Lemma 2 we need only consider three cases: $n=1$; $n=pq$ with $(p,q)=1$; and $n=p^2q$. In all other cases each sum in (7) contains only the term $|\mu(n)|$, corresponding to $d=1$, the other terms being zero. If $n=1$ the result is trivial. If $n=pq$ with $(p,q)=1$, it is easily verified that each sum in (7) is zero. In the remaining case, $n=p^2q$, each sum is equal to $-|\mu(pq)|$.

By a similar argument, Lemma 2 implies the following extension of Theorem 3.

**Theorem 4.** Let $f$ be completely multiplicative with $|f(n)| \leq 1$ for all $n$. Then for any prime $p$ and any complex $s=\sigma+it$ with $\sigma>1$ we have

$$
(1 - f(p)p^{-s}) \sum_{n=1}^{\infty} \frac{f(n)|\mu(n)|}{n^s} = (1 + f(p)p^{-s}) \sum_{n=1}^{\infty} \frac{f(n)|\mu(n)| \mu(p, n)}{n^s}.
$$

**Note.** By differentiating (4) for $\sigma>1$, letting $s \to 1+$, and using the relation ([6, §159])

$$
\sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n} = -1,
$$

we find that

$$
\sum_{n=1}^{\infty} \frac{\mu(n) \mu(p, n) \log n}{n} = \frac{p + 1}{p - 1}
$$

for every prime $p$. This implies that each of the following subseries of (8) converges to the sum indicated:

$$
\sum_{n=1; n \equiv 0 \pmod p}^{\infty} \frac{\mu(n) \log n}{n} = \frac{1}{p - 1}, \quad \sum_{n=1; n \equiv 0 \pmod p}^{\infty} \frac{\mu(n) \log n}{n} = \frac{p}{1 - p}.
$$

**References**


2. E. Landau, *Neuer Beweis der Gleichung* \( \sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0 \), *Inauguraldissertation*, Berlin, 1899.


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