MINIMAL PRIMES OF IDEALS AND INTEGRAL RING EXTENSIONS

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Abstract. It is shown that if \( R \) is a commutative ring with identity having the property that ideals in \( R \) have only a finite number of minimal primes, then a finite \( R \)-algebra again has this property. It is also shown that an almost finite integral extension of a noetherian integral domain has noetherian prime spectrum.

If \( a \) is an ideal in a ring \( R \) (tacitly assumed to be commutative with identity) and \( P \) is a prime ideal of \( R \) containing \( a \), then \( P \) is called a minimal prime of \( a \) if there is no prime ideal of \( R \) containing \( a \) and properly contained in \( P \). The ring \( R \) is said to have FC (for finite components) if each ideal of \( R \) has only a finite number of minimal primes.

I am indebted to Professor Nagata for suggestions which helped me in obtaining the proofs of the theorems in this article, and I would like to thank him for his generous help.

The special case in Theorem 1 where \( R \) is an integrally closed domain was proved in [1, Corollary 11, p. 577].

**Theorem 1.** If \( R \) is a ring with FC and \( R' \) is a finite \( R \)-algebra, then \( R' \) has FC.

**Proof.** It clearly suffices to consider the case where \( R' = R[\xi] \) is a simple \( R \)-algebra. Moreover, since FC is preserved under homomorphic image, we may assume \( R[\xi] = R[X]/(f(X)) \), where \( f(X) \) is a monic polynomial. Suppose there exists an ideal \( a' \) in \( R' \) having an infinite number of minimal primes, say \( \{P'_a\} \). We may assume that \( a' = \bigcap P'_a \). Since \( R \) has FC, \( a' \cap R = a \) has only finitely many minimal primes, so there must exist a minimal prime \( P_1 \) of \( a \) such that \( P_1 \) is contained in infinitely many of the \( P'_a \). Since \( R' = R[X]/(f(X)) \) is a finite free \( R \)-module every minimal...
prime of $P_1R'$ lies over $P_1$ in $R$ and $P_1R'$ has only finitely many minimal primes. It follows that some minimal prime $P'_1$ of $P_1R'$ is contained in infinitely many of the $P'_n$. Let $a_1 = \bigcap \{P'_a | P'_a \subseteq P'_n\}$. The fact that $P'_1$ is contained in infinitely many of the $P'_n$ implies that $a_1$ is not contained in $P'_1$ and $P'_1$ is properly contained in $a_1$. Thus $a_1$ has infinitely many minimal primes and $a_1 \cap R = a_1$ properly contains $P_1$. We note also that some minimal prime of $P_1R'$ is in the set $\{P'_a\}$. For $P_1$ occurs in a representation of $a = a' \cap R$ as a finite irredundant intersection of prime ideals. Hence if $b' = \bigcap \{P'_a | P'_a \subseteq P'_n\}$, then $b' \cap R = P_1$. Since $P_1R'$ has only finitely many minimal primes, it follows that some $P'_n$ is a minimal prime of $P_1R'$. Proceeding now in this manner with $a_1$, we can construct for any positive integer $n$, a chain $P_1 < \cdots < P_n$ of prime ideals in $R$ such that each $P_i$ is the contraction of some prime in the set $\{P'_a\}$. By choosing $n > \deg f = m$, we show that this leads to a contradiction. We may assume that $P_1 = (0)$, and hence that $R$ is an integral domain. Let $T$ be the integral closure of $R$ in an algebraic closure of the quotient field of $R$ and let $(0) = Q_1 < \cdots < Q_n$ be primes of $T$ such that $Q_i \cap R = P_i$. Let $T[\xi] = T[X]/(f(X)) = T \otimes_R R[\xi]$ and identify $T$ and $R[\xi]$ as subrings of $T \otimes_R R[\xi]$, $T = T \otimes 1$ and $R[\xi] = 1 \otimes R[\xi]$. In $T[X]$ we have $f(X) = (X - \xi_1) \cdots (X - \xi_m)$. Let $Q'_{ij}$ denote the prime in $T[\xi]$ lying over $Q_i$ in $T$ and corresponding to the root $\xi_j$ of $f(X)$. Of course not all the $Q'_{ij}$, $1 \leq j \leq m$, need be distinct, but we have $Q'_{1j} < Q'_{2j} < \cdots < Q'_{nj}$. Moreover, if $Q'_{ij} \cap R[\xi] = P'_{ij}$, then $\{P'_{ij}\}_{i=1}^n$ is the set of minimal primes of $P_iR[\xi]$ and $P'_{1j} < P'_{2j} < \cdots < P'_{nj}$. But by our construction, for each $i$, some $P'_{ij} \in \{P'_a\}$. Since there are no containment relations among the elements of $\{P'_a\}$, this implies that $n \leq m$ and completes the proof. Q.E.D.

A ring $R$ is said to have noetherian spectrum if the radical ideals in $R$ satisfy the ascending chain condition. Conditions equivalent to $R$ having noetherian spectrum are that $R$ have FC and satisfy the ascending chain condition on prime ideals.

If $R \subset R'$ are integral domains, then $R'$ is said to be almost finite over $R$ if $R'$ is integral over $R$ and if the quotient field of $R'$ is a finite algebraic extension of the quotient field of $R$ [2, p. 30].

**Theorem 2.** If $R$ is a noetherian integral domain and $R'$ is an almost finite extension of $R$, then $R'$ has noetherian spectrum.

**Proof.** It will suffice to consider the case when $R'$ is integrally closed, for if the integral closure of $R'$ has noetherian spectrum then so does $R'$. We first show that, for any ideal $a$ in $R$, $aR'$ has only a finite number of minimal primes. We proceed by induction on the number of generators for $a$. Principal ideals in $R'$ have only finitely many minimal primes, for $R'$ is the derived normal ring of a noetherian domain and hence is a
Krull domain [2, p. 118]. If $a=(x_1, \cdots, x_n)$, let $P'_1, \cdots, P'_m$ be the minimal primes of $x_1 R'$. Then $R'/P'_1$ is an almost finite extension of $R/(P'_1 \cap R)$ [2, p. 118], and by our induction assumption $a(R'/P'_1)$ has only finitely many minimal primes. Since every minimal prime of $a R'$ contains at least one of the $P'_i$, it follows that $a R'$ has only finitely many minimal primes. To complete the proof of the theorem, we can now proceed as as in Theorem 1, viz. if $a'$ were an ideal in $R'$ having infinitely many minimal primes then we could construct in $R$ an infinite strictly ascending chain $P_1 < P_2 < \cdots$ of prime ideals. This would of course contradict the fact that $R$ is noetherian.

**Corollary.** If $R$ is a noetherian integral domain, then the derived normal ring of $R$ has noetherian spectrum.

In connection with properties of the derived normal ring of a noetherian integral domain, we would like to ask the following.

**Question.** If $R$ is a noetherian integral domain with integral closure $\bar{R}$, must it follow that maximal ideals in $\bar{R}$ are finitely generated?

In trying to show this to be true, a simple induction argument on the Krull dimension of $R$ runs into the difficulty that there can exist between a 2-dimensional noetherian domain $A$ and the integral closure of $A$ a ring $B$ having a non-finitely-generated maximal ideal. To get an example illustrating this one can use the following construction suggested to me by Kaplansky. Let $T$ be a 1-dimensional local (noetherian) domain such that the integral closure of $T$ is not a finite $T$-module. Let $T < T_1 < T_2 < \cdots$ be a strictly ascending chain of finite $T$-algebras between $T$ and the integral closure of $T$. Let $X$ be an indeterminate and let $A=T[X]$, and $B=T+T_1X+T_2X^2+\cdots$. Then $B$ is a ring between $A$ and the integral closure of $A$ and if $m$ is the maximal ideal in $T$, then $m+T_1X+T_2X^2\cdots$ is a non-finitely-generated maximal ideal in $B$.

**References**


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