PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 2, October 1973

SOME ORDER AND TOPOLOGICAL PROPERTIES OF LOCALLY SOLID LINEAR TOPOLOGICAL RIESZ SPACES

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ABSTRACT. A theorem of Luxemburg and Zaanen on normed Riesz spaces (Theorem 2.4 below) and one of Nakano (Theorem 2.3 below) have been extended by the author in [1] to metrizable locally solid linear topological Riesz spaces. This note gives an example which shows they cannot be further extended to non-metrizable Hausdorff locally solid linear topological Riesz spaces.

1. **Notation and basic concepts.** For notation and terminology concerning Riesz spaces we refer to [5]. Let L be a Riesz space. A vector subspace A of L is called an ideal if $|u| \leqq |v|$ and $v \in A$ implies $u \in A$. An ideal A is called a σ-ideal if $\theta \leqq u_n \uparrow u$ in L and $\{u_n\} \subseteq A$ implies $u \in A$. An ideal A is called a band if $\theta \leqq u_\alpha \uparrow u$ in L and $\{u_\alpha\} \subseteq A$ implies $u \in A$. A subset S of L is called a solid set if $|u| \leqq |v|$ and $v \in S$ implies $u \in S$. If τ is a linear topology for L (a topology for which both mappings $(u, v) \mapsto u+v$, $(\lambda, u) \mapsto \lambda u$ are continuous), with a basis for the neighborhood system of the origin consisting of solid sets, then (L, τ) is called a locally solid linear topological Riesz space, or briefly, a locally solid Riesz space.

2. **Order and topological continuity.** Following Luxemburg and Zaanen (see [4, Notes X, XI]), we introduce the following properties for a locally solid Riesz space (L, τ).

(A, 0): $u_n \downarrow \theta$ in L and $\{u_n\}$ is a τ-Cauchy sequence implies $u_n \xrightarrow{\tau} \theta$.
(A, i): $u_n \downarrow \theta$ in L implies $u_n \xrightarrow{\tau} \theta$.
(A, ii): $u_\alpha \downarrow \theta$ in L implies $u_\alpha \xrightarrow{\tau} \theta$.
(A, iii): $\theta \leqq u_n \uparrow \leqq u$ in L, implies that $\{u_n\}$ is a τ-Cauchy sequence.
(A, iv): $\theta \leqq u_\alpha \uparrow \leqq u$ in L, implies that $\{u_\alpha\}$ is a τ-Cauchy net.

THEOREM 2.1. *Let (L, τ) be a locally solid Riesz space. Then*:
(1) (A, ii) *implies* (A, i).
(2) (L, τ) *satisfies* (A, iii) *if and only if* (L, τ) *satisfies* (A, iv).
(3) *If L is Archimedean, then* (A, ii) *implies* (A, iii).

PROOF. (1) Trivial.

Presented to the Society, January 25, 1973; received by the editors January 30, 1973.
AMS (MOS) subject classifications (1970). Primary 46A40.
Key words and phrases. Locally solid Riesz spaces, order and topological continuity.

(2) It is evident that (L, τ) satisfies (A, iii) if (L, τ) satisfies (A, iv), so we have only to show that (A, iii) implies (A, iv). Suppose that this is not so. Then there exists a net $\{u_\alpha\}$, $\theta \leqq u_\alpha \uparrow \leqq u$ in L which is not a τ-Cauchy net. This implies the existence of a circled neighborhood V of zero such that for every index α there are two indices $\alpha_1 \geqq \alpha$, $\alpha_2 \geqq \alpha$ with $u_{\alpha_1} - u_{\alpha_2} \notin V$. Now, let W be a circled neighborhood of zero such that $W + W \subseteq V$, and let α_1 be a fixed index. Then there exists an index $\alpha_2 \geqq \alpha_1$ such that $u_{\alpha_2} - u_{\alpha_1} \notin W$, otherwise $u_\alpha - u_{\alpha_1} \in W$ for all $\alpha \geqq \alpha_1$ would imply that $u_\alpha - u_{\alpha'} = (u_\alpha - u_{\alpha_1}) + (u_{\alpha_1} - u_{\alpha'}) \in W + W \subseteq V$ for all $\alpha, \alpha' \geqq \alpha_1$, contradicting the selection of the neighborhood V. Let now α_n be given. Applying the same argument we pick α_{n+1} such that $\alpha_{n+1} \geqq \alpha_n$ and with $u_{\alpha_{n+1}} - u_{\alpha_n} \notin W$. But then the sequence $\{u_{\alpha_n}\}$ satisfies $\theta \leqq u_{\alpha_n} \uparrow \leqq u$ in L and $\{u_{\alpha_n}\}$ is not a τ-Cauchy sequence. So, (L, τ) does not satisfy (A, iii), a contradiction. This completes the proof that (A, iii) implies (A, iv).

(3) Assume that L is Archimedean. Let $\theta \leqq u_n \uparrow \leqq u_0$ in L, and assume (L, τ) satisfies (A, ii). We define the set $G = \{g \in L : u_n \leqq g, \text{ for } n = 1, 2, \cdots\}$. Then we have $g - u_n \downarrow_{(g,n)} \theta$ in L (see [**5**, Theorem 22.5, p. 115]). It follows that

$$g - u_n \xrightarrow[(g,n)]{\tau} \theta,$$

and from this we easily find that $\{u_n\}$ is a τ-Cauchy sequence.

The following theorem characterizes the properties (A, i) and (A, ii) and generalizes a theorem of T. Andô and W. A. J. Luxemburg [**3**, Theorem 47.3, p. 244].

THEOREM 2.2. *Let (L, τ) be a locally solid Riesz space. Then we have*:

(i) (L, τ) *satisfies* (A, i) *if and only if every τ-closed ideal of L is a σ-ideal.*

(ii) (L, τ) *satisfies* (A, ii) *if and only if every τ-closed ideal of L is a band.*

(For a proof we refer the reader to [**1**].)

In [**4**, Note X], the following two theorems were proved.

THEOREM 2.3. *The following conditions on a normed Riesz space L_ρ are equivalent.*

(i) *L_ρ is σ-Dedekind complete, and* (A, i) *holds.*

(ii) *L_ρ is super Dedekind complete and* (A, ii) *holds.*

THEOREM 2.4. *The following conditions on a normed Riesz space L_ρ are equivalent.*

(α) *The space L_ρ satisfies both* (A, i) *and* (A, iii).

(β) *The space L_ρ satisfies* (A, ii).

Note. Theorem 2.3 is due to Nakano (see [**6**, pp. 321–322]), and Theorem 2.4 is due to Luxemburg and Zaanen (see [**4**, Theorem 33.8, Note X, p. 505]).

Theorems 2.3 and 2.4 were generalized in [**1**] for metrizable locally solid Riesz spaces.

The following example shows the above theorems do not hold for nonmetrizable Hausdorff locally solid Riesz spaces in general.

EXAMPLE 2.5. Let L be the real vector space of all real valued, Lebesgue measurable functions f defined on $[0, 1]$ such that $\int_0^1 |f(t)|^p\, dt < +\infty$, where $0<p<1$. L becomes a Riesz space under the ordering $f \leqq g$ defined by $f(x) \leqq g(x)$ for all $x \in [0, 1]$. Note that the elements of L are functions, not equivalence classes, i.e., functions differing at one point are already considered as different.

Now given $n \in N$, $\delta > 0$ and $F = \{x_1, \cdots, x_k\} \subseteqq [0, 1]$, we define the set

$$W_{F,n,\delta} = \left\{ f \in L: \int_0^1 |f(t)|^p\, dt < \frac{1}{n} \text{ and } |f(x_i)| < \delta, \text{ for } i = 1, \cdots, k \right\}.$$

As F runs over the finite subsets of $[0, 1]$, n over the natural numbers, and δ over $(0, +\infty)$ we obtain a family of sets $\{W_{F,n,\delta}\}$ which is a filter basis for a neighborhood system of the origin for a uniquely determined linear topology τ of L (note that $W_{F,2n,\delta/2} + W_{F,2n,\delta/2} \subseteqq W_{F,n,\delta}$, and that given $f \in L$, $\lambda f \in W_{F,n,\delta}$ for some $\lambda > 0$, see [**2**, p. 81]). Obviously, each $W_{F,n,\delta}$ is a solid set. It is also evident that τ is a Hausdorff topology. So, (L, τ) is a Hausdorff locally solid linear topological Riesz space. We note that (L, τ) has the following properties:

(1) τ is a sequentially complete (but not complete), nonmetrizable topology.

(2) Properties (A, i) and (A, iii) hold in (L, τ) but (A, ii) does not hold.

To see that (A, i) holds let $u_n \downarrow \theta$ in L. It is easy to show that this implies $u_n(x) \downarrow 0$ in $\boldsymbol{R}$, for all $x \in [0, 1]$. It follows now from the Lebesgue dominated convergence theorem that $\int_0^1 (u_n(t))^p\, dt \downarrow 0$. Hence $u_n \in W_{F,k,\delta}$ for all sufficiently large n. That is, $u_n \xrightarrow{\tau} \theta$.

For (A, iii), let $\theta \leqq u_n \uparrow \leqq u_0$ in L. Then $0 \leqq u_n(x) \uparrow u(x) \leqq u_0(x)$ for all $x \in [0, 1]$. It follows easily that $u \in L$ and that $\theta \leqq u_n \uparrow u$ in L. Hence

$$\begin{aligned} 0 &\leqq \int_0^1 |u_{n+k}(t) - u_n(t)|^p\, dt \\ &\leqq \int_0^1 (u(t) - u_{n+k}(t))^p\, dt + \int_0^1 (u(t) - u_n(t))^p\, dt \to 0 \end{aligned}$$

as $n, k \to +\infty$, by the Lebesgue dominated convergence theorem. From this it follows easily that $\{u_n\}$ is a τ-Cauchy sequence.

To see that (A, ii) does not hold, let $\alpha \subseteq [0, 1]$, α finite, and let $\theta \leqq u_\alpha = \chi_\alpha$. Then $\theta \leqq u_\alpha \uparrow e$ ($e(x)=1$, for all $x \in [0, 1]$), as α runs over the finite subsets of $[0, 1]$. Note that $\int_0^1 |e(t)-u_\alpha(t)|^p\, dt = 1$ for all α, and from this we see that $\{u_\alpha\}$ does not converge with respect to τ to e.

(3) L is σ-Dedekind complete, but not Dedekind complete. The σ-Dedekind completeness follows from the Lebesgue dominated convergence theorem. To see that L is not Dedekind complete, first notice that $\theta \leqq u_\alpha \uparrow u$ in L, implies $u_\alpha(x) \uparrow u(x)$ for all $x \in [0, 1]$. Now consider a nonmeasurable set E of $[0, 1]$. Define the net $\theta \leqq u_\alpha = \chi_\alpha$ for all finite subsets α of E. Then $\{u_\alpha\} \subseteq L$, $\theta \leqq u_\alpha \uparrow \leqq e$ and $u_\alpha(x) \uparrow \chi_E(x)$ for all $x \in [0, 1]$. It follows easily that $\sup\{u_\alpha\}$ does not exist in L.

(4) L satisfies (A, 0) but does not satisfy the following generalized (A, 0) property: $u_\alpha \downarrow \theta$ and $\{u_\alpha\}$ is a τ-Cauchy net implies $u_\alpha \xrightarrow{\tau} \theta$.

To see this use the net $\{u_\alpha\}$ of (2).

(5) Let $A = \{f \in L : f = 0 \text{ a.e.}\}$. Then A is a τ-closed ideal of (L, τ). Indeed if $u_\alpha \xrightarrow{\tau} u$ in L, $\{u_\alpha\} \subseteq A$, then we have in particular that

$$\int_0^1 |u(t)|^p\, dt = \int_0^1 |u_\alpha(t) - u(t)|^p\, dt \xrightarrow[(\alpha)]{} 0.$$

Thus $\int_0^1 |u(t)|^p\, dt = 0$, which shows that $u \in A$. Since $\theta \leqq u_n \uparrow u$ implies $u_n(x) \uparrow u(x)$ for all $x \in [0, 1]$ it follows easily that A is a σ-ideal. Note also that $\theta \leqq u_\alpha \uparrow e$, $\{u_\alpha\}$ is the net of (2), and $\{u_\alpha\} \subseteq A$. But $e \notin A$. This shows that A is not a band of L, in accordance with Theorem 2.2.

The following example shows that Theorems 2.3 and 2.4 are not even true for non-Hausdorff locally solid Riesz spaces having a countable basis for zero.

EXAMPLE 2.6. Consider the Riesz space L of Example 2.5 and consider the same neighborhoods $W_{F,n,\delta}$ with the restriction that F runs over all finite subsets of the rationals in $[0, 1]$. The collection $\{W_{F,n,\delta}\}$ is a filter basis for a neighborhood system of the origin for a uniquely determined linear topology τ of L. Since each $W_{F,n,\delta}$ is a solid set, (L, τ) is a locally solid linear topological Riesz space. Note that (L, τ) is non-Hausdorff having a countable basis for the neighborhoods of zero.

Using the same arguments as in Example 2.5 we can see that both (A, i) and (A, iii) hold but (A, ii) does not hold.

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