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A BOUNDED HILBERTIAN BASIS IN $C[0, 1]$

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ABSTRACT. The existence of a bounded Hilbertian basis in $C[0, 1]$ is established.

1. **Introduction.** It is well known that an orthonormal sequence $\{x_n\}$ in a Hilbert space possesses the following properties:

(1) $\sum_{i=1}^{\infty} a_i x_i$ converges implies $\sum_{i=1}^{\infty} |a_i|^2 < \infty$, and

(2) $\sum_{i=1}^{\infty} |a_i|^2$ implies $\sum_{i=1}^{\infty} a_i x_i$ converges.

A basis $\{x_i\}$ in a Banach space satisfying (1) is called a Besselian basis and a basis $\{x_i\}$ in a Banach space satisfying (2) is called an Hilbertian basis. A basis $\{x_n\}$ is bounded if $0 < \inf_n \|x_n\| \leqq \sup_n \|x_n\| < \infty$. In [2], Pełczyński raised the question whether there exists in $C[0, 1]$ (in $L^1[0, 1]$) a bounded Besselian (resp. Hilbertian) basis, in particular, whether there exists in $C[0, 1]$ a bounded orthonormal basis. In [4], Olevskiĭ proved the non-existence of bounded orthonormal basis in $C[0, 1]$; in this paper we will show that there exists a bounded Hilbertian basis in $C[0, 1]$.

A basis $\{x_n\}$ of a Banach space is wc_0 (or semishrinking) if x_n converges weakly to 0. Since every Hilbertian basis is a wc_0 basis, it is natural to look for a bounded Hilbertian basis in the class of bounded wc_0 bases. The referee has pointed out to us that Warren [3] has constructed a bounded wc_0 basis in $C[0, 1]$. Our basis is similar to his, but is a little simpler.

2. **Main result.** Let $a_0=0$, $a_1=1$ and for $j=2^n$, $a_{j+k}=(2k-1)/(2j)$ $(n=0, 1, \cdots; k=1, 2, \cdots, 2^n)$. Let $b_n=1/2^{n+1}$ $(n=1, 2, \cdots)$ and let $\{c_n\}$ be the subsequence of $\{a_n\}$ complementary to the subsequence $a_0, a_1, b_1, b_2, \cdots$ of $\{a_n\}$, i.e. $c_1=\frac{1}{2}$, $c_2=\frac{3}{4}$, $c_3=\frac{3}{8}$, $c_4=\frac{5}{8}$, $c_5=\frac{7}{8}$, $c_6=\frac{3}{16}, \cdots$. Next we define a rearrangement of $\{a_n\}$ as follows: $a_0, a_1, c_1; b_1, b_2, b_3, c_2; \cdots; b_{q(n-1)+1}, \cdots, b_{q(n)}, c_n; \cdots$, where $q(n)=1+2^2+\cdots+n^2-n$ $(n=2, 3, \cdots)$. (*Note.* To each positive integer n is associated a group of numbers consisting of n^2-1 b_i's and one c_i.) To simplify matters, let us rename the above sequence as $d_0, d_1, d_2, \cdots$.

We now define a sequence in $C[0, 1]$ as follows (it is slightly different

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from the usual way of defining a generalized Schauder basis [**1**, p. 11]):

$$\begin{aligned} x_0(t) &= 1, \\ x_1(t) &= 1 - t, \\ x_2(t) &= 1, && t = d_2, \\ &= 0, && t \in \{d_0, d_1\}, \\ &= \text{linear} && \text{for other } t, \end{aligned}$$

and for $n \geqq 3$

$$\begin{aligned} x_n(t) &= 1, && t = d_n, \\ &= 0, && t \in \{d_0, d_1, \cdots, d_{n-1}, g_n\}, \\ &= \text{linear} && \text{for other } t, \end{aligned}$$

where g_n denotes the first b_i in the sequence $\{b_i\}$ not in $\{d_0, d_1, \cdots, d_n\}$.

Finally, we define another sequence in $C[0, 1]$ as follows: $y_i = x_i$ $(i=0, 1, 2)$, and for $n \geqq 2$ and $p(n) = 1 + 2^2 + \cdots + (n-1)^2 + 1$

$$\begin{aligned} y_{p(n)+1} &= -x_{p(n)+1} - x_{p(n)+2} - \cdots - x_{p(n+1)-1} + n^{-2} x_{p(n+1)}, \\ y_{p(n)+k} &= x_{p(n)+k-1} + n^{-2} x_{p(n+1)} \qquad (k = 2, \cdots, n^2). \end{aligned}$$

We note in particular that for $n \geqq 3$, $1 \leqq k \leqq n^2$, the support of $y_{p(n)+k}$ is contained in the union of two closed disjoint intervals $I_{n,1}$ and $I_{n,2}$, where $I_{n,1} = [b_{q(n)+1}, b_{q(n-1)}]$ and $I_{n,2} = [b_{q(n-1)-1}, 1]$. (*Note.* $b_{q(n-1)-1} \leqq b_{n-1} < c_n < 1$.)

THEOREM. *The sequence $\{y_n\}$ is a bounded Hilbertian basis of $C[0, 1]$.*

PROOF. The direct method used in showing the generalized Scauder basis of $C[0, 1]$ is a basis [**1**, p. 11] can be used with slight modification to show $\{x_n\}$ is a basis. Since $x_i = y_i$ $(i=0, 1, 2)$, $x_{p(n+1)} = y_{p(n)+1} + \cdots + y_{p(n+1)}$, and $x_{p(n)+k} = y_{p(n)+k+1} - n^{-2}(y_{p(n)+1} + \cdots + y_{p(n+1)})$ $(k=1, \cdots, n^2-1)$, $[x_i] = [y_i]$ $(i=0, 1, 2)$ and $[x_{p(n)+1}, \cdots, x_{p(n+1)}] = [y_{p(n)+1}, \cdots, y_{p(n+1)}]$. Hence it suffices [**1**, p. 64] to show that there is a constant C independent of n such that for any sequence of real numbers $h_1, h_2, \cdots$,

$$\left\| \sum_{i=1}^{k} h_i y_{p(n)+i} \right\| \leqq C \left\| \sum_{i=1}^{n^2} h_i y_{p(n)+i} \right\|$$

$(n=1, 2, \cdots; k=1, 2, \cdots, n^2)$.

The following argument is similar to an argument used in [**3**]; we present it here for completeness. Let $g(k)=0$ if $k=n^2$ and $=1$ if $1 \leqq k < n^2$, then

$$\begin{aligned} &\left\| \sum_{i=1}^{k} h_i y_{p(n)+i} \right\| \\ &\qquad = \operatorname{Max}\{|h_1 + \cdots + h_k|/n^2, |h_1 - h_2|, \cdots, |h_1 - h_k|, g(k)|h_1|\} \\ &\qquad \leqq 2 \operatorname{Max}\{|h_i| \mid 1 \leqq i \leqq n^2\}. \end{aligned}$$

Now let $e_1=(h_1+\cdots+h_N)/N$, and $e_k=h_k-h_1$, $k=2, \cdots, n^2=N$; then $h_1=e_1-(e_1+e_2+\cdots+e_n)/N$, $h_k=e_k+h_1$ $(k=2, \cdots, n^2=N)$, and

$$\operatorname{Max}\{|h_i| \mid 1 \leqq i \leqq n^2\} \leqq 3 \operatorname{Max}\{|e_i| \mid 1 \leqq i \leqq n^2\}.$$

Hence for $N=n^2$

$$\begin{aligned}\left\|\sum_{i=1}^{N} h_i y_{p(n)+i}\right\| &= \operatorname{Max}\{|h_1+\cdots+h_N|/N, |h_1-h_2|, \cdots, |h_1-h_N|\} \\ &= \operatorname{Max}\{|e_i| \mid 1 \leqq i \leqq N\} \\ &\geqq \tfrac{1}{3} \operatorname{Max}\{|h_i| \mid 1 \leqq i \leqq N\} \\ &\geqq \frac{1}{6}\left\|\sum_{i=1}^{k} h_i y_{p(n)+i}\right\|.\end{aligned}$$

Hence C can be chosen to be 6.

To show that $\{y_n\}$ is a bounded Hilbertian basis, we may disregard a finite number of y_i's. Therefore, for sake of symmetry, we will consider only $y_{p(n)+k}$ for $n \geqq 3$, $1 \leqq k \leqq n^2$. Let $I_{n,1}$ and $I_{n,2}$ be the two disjoint closed intervals mentioned above. It is clear that

$$\sup\{|y_{p(n)+k}(t)| \mid t \in I_{n,1}\} = 1,$$

and

$$\sup\{|y_{p(n)+k}(t)| \mid t \in I_{n,2}\} = n^{-2} \qquad (n \geqq 3, 1 \leqq k \leqq n^2).$$

For $n \geqq 3$, $1 \leqq k \leqq n^2$, let

$$\begin{aligned} u_{p(n)+k}(t) &= y_{p(n)+k}(t), && t \in I_{n,1}, \\ &= 0 && \text{otherwise}, \\ v_{p(n)+k}(t) &= y_{p(n)+k}(t), && t \in I_{n,2}, \\ &= 0 && \text{otherwise}. \end{aligned}$$

One can readily verify that $u_{p(n)+1}=-(x_{p(n)+1}+\cdots+x_{p(n+1)-1})$, $u_{p(n)+k}=x_{p(n)+k-1}$ $(k=2, \cdots, n^2)$, and $v_{p(n)+k}=n^{-2}x_{p(n+1)}$ $(k=1, 2, \cdots, n^2)$. Thus $u_{p(n)+k}$, $v_{p(n)+k}$ are in $C[0, 1]$, $u_{p(n)+k}+v_{p(n)+k}=y_{p(n)+k}$, $\|u_{p(n)+k}\|=1$, $\|v_{p(n)+k}\|=n^{-2}$, $u_{p(n)+1} \leqq 0$ and $u_{p(n)+k} \geqq 0$ $(2 \leqq k \leqq n^2)$. And because of the way the x_i's are defined, we also have

$$\left\|\sum_{n=3}^{N} u_{p(n)+1}\right\| = \left\|\sum_{n=3}^{N} \sum_{j=1}^{n^2-1} x_{p(n)+j}\right\| = 1$$

($N \geqq 3$), and similarly,

$$\left\| \sum_{n=3}^{N-1} \sum_{k=2}^{n^2} u_{p(n)+k} + \sum_{k=2}^{K} u_{p(N)+k} \right\| = 1$$

($N \geqq 3$, $2 \leqq K \leqq N^2$).

Let $f \in C[0, 1]^*$. Then $f(y_n) = f(u_n) + f(v_n)$ ($n \geqq 7$), and

$$\sum_{n=7}^{\infty} |f(y_n)|^2 \leqq 2\left\{ \sum_{n=7}^{\infty} |f(u_n)|^2 + \sum_{n=7}^{\infty} |f(v_n)|^2 \right\}.$$

Hence it is sufficient to prove the two series on the right converge. Now $|f(v_n)| \leqq \|f\| \, \|v_n\| = \|f\|/k^2$ for some $k \geqq 3$. But because of the way the y_n's are constructed, there are exactly k^2 v_n's with $\|v_n\| = k^{-2}$. Hence

$$\sum_{n=7}^{\infty} |f(v_n)|^2 \leqq \|f\|^2 \sum_{k=3}^{\infty} k^2 \left(\frac{1}{k^2}\right)^2 < \infty.$$

To show the other series converges, we note that there exists some function h of bounded variation on $[0, 1]$ such that $f(u_n) = \int_0^1 u_n \, dh$. Let h_1 and h_2 be the positive and negative variations of h respectively. For any integer $M = p(N) + K \geqq 7$ ($N \geqq 3$, $1 \leqq K \leqq N^2$),

$$\begin{aligned}
\sum_{n=7}^{M} |f(u_n)| &= \sum_{n=3}^{N-1} \sum_{k=1}^{n^2} |f(u_{p(n)+k})| + \sum_{k=1}^{K} |f(u_{p(N)+k})| \\
&= \sum_{n=3}^{N} |f(u_{p(n)+1})| + \sum_{n=3}^{N-1} \sum_{k=2}^{n} |f(u_{p(n)+k})| + \sum_{k=2}^{K} |f(u_{p(N)+k})| \\
&\leqq \sum_{i=1}^{2} \sum_{n=3}^{N} \left| \int_0^1 u_{p(n)+1} \, dh_i \right| \\
&\quad + \sum_{i=1}^{2} \left\{ \sum_{n=3}^{N-1} \sum_{k=2}^{n^2} \left| \int_0^1 u_{p(n)+k} \, dh_i \right| + \sum_{k=2}^{K} \left| \int_0^1 u_{p(N)+k} \, dh_i \right| \right\} \\
&= \sum_{i=1}^{2} \left| \int_0^1 \left(\sum_{n=3}^{N} u_{p(n)+1} \right) dh_i \right| \\
&\quad + \sum_{i=1}^{2} \left| \int_0^1 \left(\sum_{n=3}^{N-1} \sum_{k=2}^{n^2} u_{p(n)+k} + \sum_{k=2}^{K} u_{p(N)+k} \right) dh_i \right| \\
&\leqq 2(V(h_1) + V(h_2))
\end{aligned}$$

where $V(h_i)$ ($i = 1, 2$) are the total variations of the monotone increasing functions h_i. This obviously implies the convergence of the series.

Finally, the boundedness is clear. This completes the proof.

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