CERTAIN SUBSETS OF PRODUCTS OF $\theta$-REFINABLE SPACES ARE REALCOMPACT

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Abstract. It is shown that the normal $T_1$-space $X$ is realcompact if and only if (a) each discrete subset of $X$ is realcompact and (b) $X$ can be embedded as a closed subset in the product of a collection of regular $\theta$-refinable spaces.

We will say that a space $X$ has property (*) if it is true that each discrete subset of $X$ is realcompact; i.e., the cardinality of each discrete subset of $X$ is nonmeasurable. In [5], the author has shown that a normal $T_1$-space $X$ is realcompact if and only if $X$ has property (*) and $X$ can be embedded as a closed subspace in the product of a collection of subparacompact spaces and metacompact spaces. S. Mrówka suggested to the author that there should be a nontrivial class of spaces $\mathcal{P}$ containing the class of subparacompact spaces and the class of metacompact spaces so that a normal space $X$ is realcompact if and only if $X$ has property (*) and $X$ can be embedded as a closed subspace in a product of members of $\mathcal{P}$. It is the purpose of this paper to show that the class of $\theta$-refinable spaces, introduced by Worrell and Wicke in [4], is such a class.

Recall that a space $X$ is $\theta$-refinable if it is true that if $\mathcal{V}$ is an open cover of $X$ then there is a sequence $\mathcal{V}_1, \mathcal{V}_2, \cdots$ of open covers of $X$ that refine $\mathcal{V}$ such that if $x \in X$, then there is an integer $i$ such that only finitely many members of $\mathcal{V}_i$ contain $x$. Clearly, any metacompact space is $\theta$-refinable. It is shown in [1] that any subparacompact space is $\theta$-refinable.

Our notation will follow that of [2].

Lemma 1 [5]. Suppose that $X$ is a $T_1$-space and $\mathcal{E}$ is a class of $T_3$-spaces such that the topology on $X$ is the weak topology induced by $C(X, \mathcal{E})$. Then $X$ can be embedded as a closed subspace in the product of a collection of members of $\mathcal{E}$ if and only if it is true that if $\mathcal{F}$ is a free ultrafilter of closed subsets of $X$, then there are a member $f$ of $C(X, \mathcal{E})$ and an open cover $\mathcal{U}$ of range $(f)$ such that $\{f^{-1}(U) : U \in \mathcal{U}\}$ refines $\{(X-F) : F \in \mathcal{F}\}$.

Lemma 2 (Theorem 18, [3]). If $\mathcal{U}$ is an open cover of the space $X$, then
there is a discrete subspace \( H \) of \( X \) such that

(i) \( \{ \text{st}(x, U) : x \in H \} \) covers \( X \), and

(ii) no member of \( \mathcal{U} \) contains two points of \( H \).

**Theorem.** The following conditions on a normal \( T_1 \)-space \( X \) are equivalent:

1. \( X \) is realcompact.
2. \( X \) has property (*) and \( X \) can be embedded as a closed subset in the product of a collection of regular \( \theta \)-refinable spaces.
3. \( X \) has property (*) and if \( \mathcal{F} \) is a free ultrafilter of closed subsets of \( X \), then there is a sequence \( \mathcal{W}_1, \mathcal{W}_2, \ldots \) of open covers of \( X \) refining \( \{ X - F : F \in \mathcal{F} \} \) such that if \( x \in X \) then there is an integer \( i \) such that only finitely many members of \( \mathcal{W}_i \) contain \( x \).

**Proof.** (1) implies (2). This is obvious since every closed subset of a realcompact space is realcompact and the real line is \( \theta \)-refinable.

(2) implies (3). Let \( \mathcal{F} \) be a free ultrafilter of closed subsets of \( X \). According to Lemma 1, there are a \( \theta \)-refinable space \( Y \), an open cover \( \mathcal{V} \) of \( Y \), and a continuous function \( f \) taking \( X \) into \( Y \) such that \( f^{-1}(\mathcal{V}) = \{ f^{-1}(V) : V \in \mathcal{V} \} \) refines \( \{ X - F : F \in \mathcal{F} \} \). Since \( Y \) is \( \theta \)-refinable, there is a sequence \( \mathcal{V}_1, \mathcal{V}_2, \ldots \) of open covers of \( Y \) refining \( \mathcal{V} \) such that if \( y \in Y \), there is an integer \( i \) such that only finitely many members of \( \mathcal{V}_i \) contain \( y \). Clearly, if for each \( i \), \( \mathcal{W}_i = f^{-1}(\mathcal{V}_i) \), the sequence \( \mathcal{W}_1, \mathcal{W}_2, \ldots \) satisfies condition (3) of our theorem.

(3) implies (1). Suppose that \( X \) satisfies condition (3) but \( X \) is not realcompact. Let \( \mathcal{L} \) be a free \( \mathbb{Z} \)-ultrafilter in \( X \) with the countable intersection property. Let \( \mathcal{F} \) be the ultrafilter of closed subsets of \( X \) that contains \( \mathcal{L} \) (\( \mathcal{F} \) is uniquely determined by \( \mathcal{L} \) since \( X \) is normal). Let \( \mathcal{W}_1, \mathcal{W}_2, \ldots \) be a sequence of open covers of \( X \) refining \( \{ X - F : F \in \mathcal{F} \} \) such that if \( x \in X \) then there is an \( i \) such that only finitely many members of \( \mathcal{W}_i \) contain \( x \). For each pair \((i, j)\) of positive integers, let \( H(i, j) = \{ x \in X : x \text{ is contained in at most } j \text{ members of } \mathcal{W}_i \} \). It is easy to see that each \( H(i, j) \) is closed. Let \( \mathcal{H} \) denote collection of all \( H(i, j) \)'s. Let \( \mathcal{H}_x = \mathcal{H} - \mathcal{F} \) and \( \mathcal{H}_x = \mathcal{H} - \mathcal{H}_1 \). For each \( H \) in \( \mathcal{H}_1 \), let \( F(H) \) denote a member of \( \mathcal{F} \) that does not intersect \( H \). Since \( X \) is normal, there is a zero-set \( Z(H) \) containing \( F(H) \) that does not intersect \( H \). For each \( H \) in \( \mathcal{H}_1 \), \( Z(H) \) is in \( \mathcal{L} \). It must be the case that \( \mathcal{H}_2 \) is not empty; otherwise, \( \{ Z(H) : H \in \mathcal{H}_1 \} \) would be a countable subcollection of \( \mathcal{L} \) with no common part which would be a contradiction.

For each \( H = H(i, j) \) in \( \mathcal{H}_2 \), there is, by Lemma 2, a discrete subset \( K(H) \) of \( H \) such that no member of \( \mathcal{W}_i \) contains two members of \( K(H) \) and \( \{ \text{st}(x, W_i) : x \in K(H) \} \) covers \( H \). Note that \( K(H) \) is infinite for otherwise, the collection \( \{ W \in \mathcal{W}_i : W \cap K(H) \neq \emptyset \} \) would be finite and \( \bigcap \{ X - W : W \in \mathcal{W}_i, W \cap K(H) \neq \emptyset \} \) would be a member of \( \mathcal{F} \) that would
not intersect $H$ and this would contradict the assumption that $H \in \mathcal{H}_2$. Let $\mathcal{W}' = \{ W \in \mathcal{W} : W \cap K(H) \neq \emptyset \}$. Since $K(H)$ is infinite and each point of $H$ is contained in only finitely many members of $\mathcal{W}'$, it must be true that the cardinality of $K(H)$ is the same as the cardinality of $\mathcal{W}'$. Let $\varphi$ be a one-to-one function from $K(H)$ onto $\mathcal{W}'$. For each $F$ in $\mathcal{F}$, let $M(F) = \{ x \in K(H) : \varphi(x) \cap (F \cap H) \neq \emptyset \}$. Clearly, $\{ M(F) : F \in \mathcal{F} \}$ has the finite intersection property; and so, there is an ultrafilter $\mathcal{M}$ of subsets of $K(H)$ that contains $\{ M(F) : F \in \mathcal{F} \}$. Since, for each $x \in K(H)$, it is true that $X - \varphi(x) \in \mathcal{F}$, it is true that $\mathcal{M}$ is a free ultrafilter of subsets of $K(H)$. Since $K(H)$ is a discrete subset of $X$, $K(H)$ is realcompact; and so, there is a countable subcollection $\{ M_i \}$ of members of $\mathcal{M}$ with no common part.

Claim 1. If $M \in \mathcal{M}$, there is a member $F$ of $\mathcal{F}$ that is a subset of $\bigcup_{x \in M} \varphi(x)$.

The argument for this is the same as the argument for Claim 1 in the proof of the theorem in [5].

Claim 2. $[\bigcap_{i=1}^{\infty} (\bigcup_{x \in M_i} (\varphi(x)))] \cap H = \emptyset$.

Again, the argument for this is the same as the argument for Claim 2 in the proof of the theorem in [5].

By Claim 1, for each integer $n$, there is a member $F_n$ of $\mathcal{F}$ such that $F_n \subseteq \bigcup_{x \in M_n} (\varphi(x))$. Since $X$ is normal, there is a zero-set $Z_n$ such that $F_n \subseteq Z_n \subseteq \bigcup_{x \in M_n} (\varphi(x))$. It follows from Claim 2 that $\bigcap (Z_n \cap H) = \emptyset$. Thus, for each $H \in \mathcal{H}_2$ there is a countable subcollection $\mathcal{L}(H)$ of $\mathcal{L}$ such that $[\bigcap_{Z \in \mathcal{L}(H)} (Z)] \cap H = \emptyset$. Thus, we have $\{ Z(H) : H \in \mathcal{H}_2 \} \cup (\bigcup_{H \in \mathcal{H}_2} \mathcal{L}(H))$ is a countable subcollection of $\mathcal{L}$ with no common part which contradicts the assumption that $\mathcal{L}$ has the countable intersection property.

Note. In [5], the author asked if every normal metacompact space is topologically complete (in the sense of Dieudonné). R. Haydon offers an example of a normal metacompact space which is not complete in [6].

References