CERTAIN SUBSETS OF PRODUCTS OF $\theta$-REFINABLE SPACES ARE REALCOMPACT

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Abstract. It is shown that the normal $T_1$-space $X$ is realcompact if and only if (a) each discrete subset of $X$ is realcompact and (b) $X$ can be embedded as a closed subset in the product of a collection of regular $\theta$-refinable spaces.

We will say that a space $X$ has property (*) if it is true that each discrete subset of $X$ is realcompact; i.e., the cardinality of each discrete subset of $X$ is nonmeasurable. In [5], the author has shown that a normal $T_1$-space $X$ is realcompact if and only if $X$ has property (*) and $X$ can be embedded as a closed subspace in the product of a collection of subparacompact spaces and metacompact spaces. S. Mrówka suggested to the author that there should be a nontrivial class of spaces $\mathcal{P}$ containing the class of subparacompact spaces and the class of metacompact spaces so that a normal space $X$ is realcompact if and only if $X$ has property (*) and $X$ can be embedded as a closed subspace in a product of members of $\mathcal{P}$. It is the purpose of this paper to show that the class of $\theta$-refinable spaces, introduced by Worrell and Wicke in [4], is such a class.

Recall that a space $X$ is $\theta$-refinable if it is true that if $\mathcal{V}$ is an open cover of $X$ then there is a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of open covers of $X$ that refine $\mathcal{V}$ such that if $x \in X$, then there is an integer $i$ such that only finitely many members of $\mathcal{V}_i$ contain $x$. Clearly, any metacompact space is $\theta$-refinable. It is shown in [1] that any subparacompact space is $\theta$-refinable.

Our notation will follow that of [2].

Lemma 1 [5]. Suppose that $X$ is a $T_1$-space and $\mathcal{E}$ is a class of $T_3$-spaces such that the topology on $X$ is the weak topology induced by $C(X, \mathcal{E})$. Then $X$ can be embedded as a closed subspace in the product of a collection of members of $\mathcal{E}$ if and only if it is true that if $\mathcal{F}$ is a free ultrafilter of closed subsets of $X$, then there are a member $f$ of $C(X, \mathcal{E})$ and an open cover $\mathcal{U}$ of range $(f)$ such that $f^{-1}(U) : U \in \mathcal{U}$ refines $(X - F) : F \in \mathcal{F}$.

Lemma 2 (Theorem 18, [3]). If $\mathcal{U}$ is an open cover of the space $X$, then

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there is a discrete subspace $H$ of $X$ such that

(i) $\{st(x, U) : x \in H\}$ covers $X$, and

(ii) no member of $\mathcal{U}$ contains two points of $H$.

**Theorem.** The following conditions on a normal $T_1$-space $X$ are equivalent:

1. $X$ is realcompact.
2. $X$ has property (*) and $X$ can be embedded as a closed subset in the product of a collection of regular $\theta$-refinable spaces.
3. $X$ has property (*) and if $\mathcal{F}$ is a free ultrafilter of closed subsets of $X$, then there is a sequence $\mathcal{W}_1, \mathcal{W}_2, \cdots$ of open covers of $X$ refining $\{X-F : F \in \mathcal{F}\}$ such that if $x \in X$ then there is an integer $i$ such that only finitely many members of $\mathcal{W}_i$ contain $x$.

**Proof.** (1) implies (2). This is obvious since every closed subset of a realcompact space is realcompact and the real line is $\theta$-refinable.

(2) implies (3). Let $\mathcal{F}$ be a free ultrafilter of closed subsets of $X$. According to Lemma 1, there are a $\theta$-refinable space $Y$, an open cover $\mathcal{V}$ of $Y$, and a continuous function $f$ taking $X$ into $Y$ such that $f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$ refines $\{X-F : F \in \mathcal{F}\}$. Since $Y$ is $\theta$-refinable, there is a sequence $\mathcal{V}_1, \mathcal{V}_2, \cdots$ of open covers of $Y$ refining $\mathcal{V}$ such that if $y \in Y$, there is an integer $i$ such that only finitely many members of $\mathcal{V}_i$ contain $x$. Clearly, if for each $i$, $\mathcal{W}_i = f^{-1}(\mathcal{V}_i)$, the sequence $\mathcal{W}_1, \mathcal{W}_2, \cdots$ satisfies condition (3) of our theorem.

(3) implies (1). Suppose that $X$ satisfies condition (3) but $X$ is not realcompact. Let $\mathcal{L}$ be a free $Z$-ultrafilter in $X$ with the countable intersection property. Let $\mathcal{F}$ be the ultrafilter of closed subsets of $X$ that contains $\mathcal{L}$ ($\mathcal{F}$ is uniquely determined by $\mathcal{L}$ since $X$ is normal). Let $\mathcal{W}_1, \mathcal{W}_2, \cdots$ be a sequence of open covers of $X$ refining $\{X-F : F \in \mathcal{F}\}$ such that if $x \in X$ then there is an $i$ such that only finitely many members of $\mathcal{W}_i$ contain $x$. For each pair $(i,j)$ of positive integers, let $H(i,j) = \{x \in X : x \text{ is contained in at most } j \text{ members of } \mathcal{W}_i\}$. It is easy to see that each $H(i,j)$ is closed. Let $\mathcal{H}$ denote collection of all $H(i,j)$'s. Let $\mathcal{H}_1 = \mathcal{H} - \mathcal{F}$ and $\mathcal{H}_2 = \mathcal{H} - \mathcal{H}_1$. For each $H$ in $\mathcal{H}_1$, let $F(H)$ denote a member of $\mathcal{F}$ that does not intersect $H$. Since $X$ is normal, there is a zero-set $Z(H)$ containing $F(H)$ that does not intersect $H$. For each $H$ in $\mathcal{H}_1$, $Z(H)$ is in $\mathcal{L}$. It must be the case that $\mathcal{H}_2$ is not empty; otherwise, $\{Z(H) : H \in \mathcal{H}_1\}$ would be a countable subcollection of $\mathcal{L}$ with no common part which would be a contradiction.

For each $H = H(i,j)$ in $\mathcal{H}_2$, there is, by Lemma 2, a discrete subset $K(H)$ of $H$ such that no member of $\mathcal{W}_i$ contains two members of $K(H)$ and $\{st(x, W_i) : x \in K(H)\}$ covers $H$. Note that $K(H)$ is infinite for otherwise, the collection $\{W \in \mathcal{W}_i : W \cap K(H) \neq \emptyset\}$ would be finite and $\bigcap \{X-W : W \in \mathcal{W}_i, W \cap K(H) \neq \emptyset\}$ would be a member of $\mathcal{F}$ that would
not intersect \( H \) and this would contradict the assumption that \( H \in \mathcal{H}_2 \). Let \( \mathcal{W}_i = \{ W \in \mathcal{W} : W \cap K(H) \neq \emptyset \} \). Since \( K(H) \) is infinite and each point of \( H \) is contained in only finitely many members of \( \mathcal{W}_i \), it must be true that the cardinality of \( K(H) \) is the same as the cardinality of \( \mathcal{W}_i \). Let \( \varphi \) be a one-to-one function from \( K(H) \) onto \( \mathcal{W}_i \). For each \( F \) in \( \mathcal{F} \), let \( M(F) = \{ x \in K(H) : \varphi(x) \cap (F \cap H) \neq \emptyset \} \). Clearly, \( \{ M(F) : F \in \mathcal{F} \} \) has the finite intersection property; and so, there is an ultrafilter \( \mathcal{M} \) of subsets of \( K(H) \) that contains \( \{ M(F) : F \in \mathcal{F} \} \). Since, for each \( x \in K(H) \), it is true that \( X - \varphi(x) \in \mathcal{F} \), it is true that \( \mathcal{M} \) is a free ultrafilter of subsets of \( K(H) \). Since \( K(H) \) is a discrete subset of \( X \), \( K(H) \) is realcompact; and so, there is a countable subcollection \( \{ M_i \} \) of members of \( \mathcal{M} \) with no common part.

**Claim 1.** If \( M \in \mathcal{M} \), there is a member \( F \) of \( \mathcal{F} \) that is a subset of \( \bigcup_{x \in M} \varphi(x) \).

The argument for this is the same as the argument for Claim 1 in the proof of the theorem in [5].

**Claim 2.** \( \bigcap_{i=1}^{\infty} \left( \bigcup_{x \in M_i} \varphi(x) \right) \cap H = \emptyset \).

Again, the argument for this is the same as the argument for Claim 2 in the proof of the theorem in [5].

By Claim 1, for each integer \( n \), there is a member \( F_n \) of \( \mathcal{F} \) such that \( F_n \subseteq \bigcup_{x \in M_n} \varphi(x) \). Since \( X \) is normal, there is a zero-set \( Z_n \) such that \( F_n \subseteq Z_n \subseteq \bigcup_{x \in M_n} \varphi(x) \). It follows from Claim 2 that \( \bigcap (Z_n \cap H) = \emptyset \). Thus, for each \( H \in \mathcal{H}_2 \) there is a countable subcollection \( \mathcal{L}(H) \) of \( \mathcal{L} \) such that \( \left( \bigcap_{x \in \mathcal{L}(H)} Z \right) \cap H = \emptyset \). Thus, we have \( \{ Z(H) | H \in \mathcal{H}_1 \} \cup \left( \bigcup_{H \in \mathcal{H}_2} \mathcal{L}(H) \right) \) is a countable subcollection of \( \mathcal{L} \) with no common part which contradicts the assumption that \( \mathcal{L} \) has the countable intersection property.

**Note.** In [5], the author asked if every normal metacompact space is topologically complete (in the sense of Dieudonné). R. Haydon offers an example of a normal metacompact space which is not complete in [6].

**References**


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