A PRIME-DIVISOR FUNCTION

J. KNOPFMACHER

Abstract. This note studies the asymptotic mean values over arithmetical progressions, the general distribution of values, and the maximum order of magnitude, of a certain natural prime-divisor function of positive integers.

Consider the multiplicative arithmetical function $\beta$ defined by $\beta(1) = 1$ and $\beta(n) = \alpha_1 \alpha_2 \cdots \alpha_r$ if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ (prime, $\alpha_i > 0$). Kendall and Rankin [2, p. 199] pointed out that this function has the finite mean value

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \beta(n) = \frac{\zeta(2) \zeta(3)}{\zeta(6)} = 1.943 \ldots
$$

Strangely, perhaps, there appears to be virtually no other information available about this natural arithmetical function. (See note added in proof.) This note makes a more detailed study of its asymptotic properties.

1. Average values and distribution.

Theorem 1. Let $r$ and $q$ denote relatively prime positive integers. Then,

$$
\sum_{n \leq x; n \equiv r \pmod{q}} \beta(n) = \frac{1}{q} \left( L(2, \chi_0) L(3, \chi_0) x + \frac{L(\frac{5}{2}, \chi_0) L(\frac{3}{2}, \chi_0)}{L(3, \chi_0)} x^{1/2} \right.
$$

$$
+ \frac{L(\frac{3}{2}, \chi_0) L(\frac{5}{2}, \chi_0)}{L(2, \chi_0)} x^{1/2} + \frac{\chi_1(r) L(\frac{1}{2}, \chi_1) L(\frac{1}{2}, \chi_1)}{L(3, \chi_0)} x^{1/2} \left.ight)
$$

$$
+ \sum_{\chi_2} \frac{\chi_2(r) L(\frac{1}{2}, \chi_2) L(\frac{1}{2}, \chi_2)}{L(2, \chi_0)} x^{1/2} \right)
$$

$$
+ O(x^{3/10} \log^{9/10} x \phi(q)^{2/5}),
$$

where $\chi_0$ denotes the principal character mod $q$, and the terms in $\chi_1$, $\chi_2$ occur if and only if there exist characters $\chi_1 \neq \chi_0$, $\chi_2 \neq \chi_0$ mod $q$ such that $\chi_1^2 = \chi_0$, $\chi_2^3 = \chi_0$.

Received by the editors November 21, 1972.


Key words and phrases. Asymptotic mean value, arithmetical progression, asymptotic distribution function, frequency, maximum order of magnitude.
PROOF. Given any character $\chi \bmod q$, the Euler product formula for $L(z, \chi)$ leads to the equation

$$
\sum_{n=1}^{\infty} \chi(n) \beta(n)n^{-z} = \frac{L(z, \chi)L(2z, \chi^2)L(3z, \chi^3)}{L(6z, \chi^6)} \quad \text{[Re } z > 1\text{].}
$$

Therefore, if $L(z, \chi)L(2z, \chi^2)L(3z, \chi^3) = \sum_{n=1}^{\infty} c(n, \chi)n^{-z} \quad [\text{Re } z > 1]$, then

$$
\sum_{n \leq x} \chi(n) \beta(n) = \sum_{k \leq x} \phi(q) x^{1/2} L(\frac{1}{2}, \chi_1) L(\frac{3}{2}, \chi_1) + \sum_{m \leq x^{3/10}} \chi^6(m) \mu(m) \sum_{k \leq x/m^4} c(k, \chi).
$$

Now, by equations (68–71) of Richert [7], we have:

\begin{enumerate}
\item \[
\sum_{n \leq x} c(n, \chi_0) = \frac{\phi(q)}{q} \left\{ xL(2, \chi_0)L(3, \chi_0) + x^{1/2} L(\frac{1}{2}, \chi_0)L(\frac{3}{2}, \chi_0) \right\} + O(x^{3/10} \log^{9/10} x \phi(q)^{9/5});
\]
\item \[
\sum_{n \leq x} c(n, \chi_1) = \frac{\phi(q)}{q} x^{1/2} L(\frac{1}{2}, \chi_1)L(\frac{3}{2}, \chi_1) + O(x^{3/10} \log^{9/10} x \phi(q)^{4/3}) \quad \text{if } \chi_1^2 = \chi_0 \text{ but } \chi_1 \neq \chi_0;
\]
\item \[
\sum_{n \leq x} c(n, \chi_2) = \frac{\phi(q)}{q} x^{1/2} L(\frac{1}{2}, \chi_2)L(\frac{3}{2}, \chi_2) + O(x^{3/10} \log^{9/10} x \phi(q)^{7/5}) \quad \text{if } \chi_2^3 = \chi_0 \text{ but } \chi_2 \neq \chi_0;
\]
\item \[
\sum_{n \leq x} c(n, \chi) = O(x^{3/10} \log^{9/10} x \phi(q)^{8/3}) \quad \text{for all other } \chi.
\]
\end{enumerate}

It follows for example that

$$
\sum_{n \leq x} \chi_1(n) \beta(n) = \sum_{m \leq x^{3/10}} \chi_1^6(m) \mu(m) \left( \frac{\phi(q)x^{1/2}}{qm^3} L(\frac{1}{2}, \chi_1)L(\frac{3}{2}, \chi_1) \right)
$$

$$
+ O((x/m^6)^{3/10} \log^{9/10}(x/m^6) \phi(q)^{4/3})
$$

$$
= \frac{\phi(q)}{q} x^{1/2} L(\frac{1}{2}, \chi_1)L(\frac{3}{2}, \chi_1) \left[ \frac{1}{L(3, \chi_0)} + O(x^{-1/3}) \right]
$$

$$
+ O(x^{3/10} \log^{9/10} x \phi(q)^{4/3})
$$

$$
= \frac{\phi(q)}{q} x^{1/2} \frac{L(\frac{1}{2}, \chi_1)L(\frac{3}{2}, \chi_1)}{L(3, \chi_0)} + O(x^{3/10} \log^{9/10} x \phi(q)^{4/3}).
$$

Similarly one can determine estimates for $\sum_{n \leq x} \chi(n) \beta(n)$ in the other cases. The theorem then follows from the equation

$$
\sum_{n \leq x; n \equiv r \pmod q} \beta(n) = \sum_{n \leq x} \beta(n) \frac{1}{\phi(q)} \sum_{\chi \bmod q} \chi(n) \tilde{\chi}(r).
$$
Corollary.  
\[
\sum_{n \leq x} \beta(n) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} x + \frac{\zeta(4)\zeta(3)}{\zeta(5)} x^{1/2} + \frac{\zeta(4)\zeta(3)}{\zeta(2)} x^{1/3} + O(x^{3/10} \log^{9/10} x).
\]

Remark. By using a theorem of Schmidt [5] concerning \( \sum_{n \leq x} c(n) \) where \( \sum c(n)n^{-z} = \zeta(z)\zeta(2z)\zeta(3z) \) [Re \( z > 1 \)], one can sharpen the error term in this corollary to \( O(x^{7/27} \log^2 x) \). For a more general, though less sharp, version of the corollary, see the author [3]. (See note added in proof.)

Next, by applying a theorem of Schoenberg [6, p. 319], one obtains

Theorem 2. The function \( \beta \) possesses an asymptotic distribution function

\[
F(x) = \lim_{N \to \infty} \frac{1}{N} \text{card}\{n \leq N : \beta(n) \leq x\}.
\]

This function is discrete, and the characteristic function of \( F(e^t) \) is

\[
\int_{-\infty}^{\infty} e^{itx} \, dF(e^t) = \prod_{\text{primes } p} \left( 1 + \sum_{r=1}^{\infty} p^{-r}e^{it\log r} \right).
\]

Remark. As part of a different discussion, J. Ridley and the author [4] have shown that, for each \( k = 1, 2, \ldots \), the function \( \beta \) has a finite \( k \)th moment

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} [\beta(n)]^k = \prod_{\text{primes } p} \left\{ 1 + \sum_{r=2}^{\infty} [r^k - (r - 1)^k] p^{-r} \right\}.
\]

It may also be mentioned that a slight modification of a technique of Kendall and Rankin [2, p. 204] (who are concerned with the total number \( a(n) \) of nonisomorphic abelian groups of order \( n \)) leads to an explicit formula for the frequency

\[
F_m = \lim_{N \to \infty} \frac{1}{N} \text{card}\{n \leq N : \beta(n) = m\}.
\]

Since this formula is relatively involved, we do not give it in detail, but we note that by combining certain of the frequencies \( P_m \) calculated for \( a(n) \) in [2, p. 205] one obtains:

\[
F_1 = P_1 = 6/\pi^2 = 0.6079 \ldots, \quad F_2 = P_2 = 0.2008 \ldots, \\
F_3 = P_3 = 0.0742 \ldots, \quad F_4 = P_4 + P_5 = 0.0542 \ldots, \\
F_5 = P_7 = 0.0147 \ldots, \quad F_6 = P_6 + P_{11} = 0.0215 \ldots.
\]

Thus \( F_1 + \cdots + F_8 = 0.9733 \ldots \), which emphasizes how closely the values of \( \beta \) cluster about its mean value 1.943 \ldots. In fact, after a computer
check on the “empirical” frequencies of $\beta$ over the range $1 \leq n \leq 10,000$, J. Ridley has very kindly provided the following figures for the actual frequencies in this range: $F'_1 = 0.6083, F'_2 = 0.2008, F'_3 = 0.0744, F'_4 = 0.0541, F'_5 = 0.0151, F'_6 = 0.0216$. Here $F'_1 + \cdots + F'_6 = 0.9743$.

2. Maximum order of magnitude.

**Theorem 3.** Given any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that

$$\beta(n) < 3^{(1/3)(1+\varepsilon)\log n / \log \log n} \text{ for all } n \geq n_0(\varepsilon),$$

while

$$\beta(n) > 3^{(1/3)(1-\varepsilon)\log n / \log \log n} \text{ for infinitely many } n.$$

**Proof.** The argument is parallel to one whereby Hardy and Wright [1] proved an analogous theorem for the divisor function $d(n)$. Firstly, one notes by induction that $a \leq 3^{a/3}$ for $a = 1, 2, \ldots$. Hence, for $p \geq 3^{1/3\delta}$ ($\delta > 0$) and $a = 1, 2, \ldots$,

$$\frac{a}{p^{\delta a}} \leq \frac{a}{3^{a/3}} \leq 1.$$

If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ ($p_i$ prime, $a_i > 0$), it follows that

$$\beta(n) = \prod_{i=1}^{r} \frac{a_i}{p_i^{\delta a_i}} \leq \prod_{\text{primes } p \leq 3^{1/3\delta}} (\delta \log 2)^{-1} \exp\left(\frac{3^{1/3\delta}}{\delta \log 2}\right).$$

If $\delta = (1 + \frac{1}{2} \varepsilon) \log 3/3 \log \log n$ ($\varepsilon > 0$), then

$$\frac{3^{1/3\delta}}{\delta \log 2} = \frac{(\log n)^{1/(1+\varepsilon/2)} \log \log n}{(1 + \frac{1}{2} \varepsilon) \log 2 \cdot \log 3^{1/3}} < \varepsilon \log 3^{1/3} \cdot \log n \cdot \frac{\log \log n}{2 \log \log n}$$

for $n$ sufficiently large. Hence the upper inequality follows.

For the lower inequality, let $N = (p_1 p_2 \cdots p_r)^3$ where $p_1 < \cdots < p_r$ denote the first $r$ primes. Then

$$\log \beta(N) = r \log 3 = \pi(p_r) \log 3 \geq \frac{\log(p_1 p_2 \cdots p_r) \cdot \log 3}{\log p_r},$$

in a similar way to the situation in [1, p. 263]. Hence, in the same way, there is a constant $C$ such that

$$\log \beta(N) \geq \frac{\log N \cdot \log 3^{1/3}}{\log \log N + C} > \frac{(1 - \varepsilon) \log 3^{1/3} \cdot \log N}{\log \log N}$$

for $N$ sufficiently large, i.e. for $r$ sufficiently large.

**Theorem 4.** Given any $\varepsilon > 0$, $\beta(n) < 3^{(1/3)(1+\varepsilon) \log \log n}$ for "almost all" $n$, i.e. all $n$ outside some set of asymptotic density zero.
Proof. The inequality $a \leq 3^{a/3} (a = 1, 2, \cdots)$ implies that $\beta(n) \leq 3^{\Omega(n)/3}$ where $\Omega(n)$ is the sum of the exponents of the prime divisors of $n$. The theorem then follows from Theorem 431 of [1], which states that $\Omega(n)$ has "normal order" $\log \log n$.

Remark. One cannot expect a similar lower inequality, since for example $\beta$ takes the value 1 on all square-free integers, and these have positive density $6/\pi^2$.

Note added in proof. In a recent paper, The number of square-full divisors of an integer, Proc. Amer. Math. Soc. 34 (1972), 79–80, D. Suryanarayana and R. Sita Rama Chandra Rao established the above corollary to Theorem 1, with an error estimate slightly weaker than that obtainable with the aid of Schmidt's theorem [5].

References


Department of Mathematics, University of the Witwatersrand, Johannesburg, South Africa