ON RETRACEABLE SETS WITH RAPID GROWTH

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Abstract. We combine a refinement of a recent theorem of A. N. Degtev with a result of our own, in order to derive a general theorem about regressive sets which has the following corollary. If $A$ is any point-decomposable $\pi^0_1$ set then $A$ has an infinite $\pi^0_1$ subset $B$ such that $B$ has "highly" dense-simple complement and, moreover, all infinite $\pi^0_1$ subsets of $B$ are effectively decomposable in a strong sense (namely, they are all retraceable).

1. Introduction and principal theorem. Various recent articles ([1], [3], [4], [10], and, implicitly, [9]) have dealt with (infinite) retraceable sets $A$ having the following property: if $p_A$ is the principal function of $A$ (i.e., the function which enumerates $A$ in order of magnitude) and if $q_\varepsilon$ is any partial recursive function, then $q_\varepsilon(p_A(n)) < p_A(n+1)$ holds for almost all $n$. Let us refer to this phenomenon as property (P), independently of whether $A$ is retraceable. In [10], we proved some general theorems about regressing functions which immediately imply the following result:

Theorem 1 (cf. [10, Theorems 31.1 and 31.2]). If $A$ is any infinite regressive set of natural numbers, then there exist sets $B$ and $C$ such that $C$ is r.e., $B = A \cap C$, and $B$ is a retraceable set having both property (P) and, also, the property (which we shall call property (Q)) that $p_B(n) > q_\varepsilon(n)$ holds for all sufficiently large $n$, for any partial recursive function $q_\varepsilon$. (As is noted in [10], it is in fact the case that $(P) \Rightarrow (Q)$ holds for all retraceable sets.)

Naturally, we refer in both (P) and (Q) only to those $x$ for which $q_\varepsilon(x)$ is defined.

(For background information on retraceable and regressive sets, the reader may consult [2].)

For the convenience of the reader, we shall briefly (and, in the case of Theorem 31.1, very informally) indicate the content of Theorems 31.1

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and 31.2 of [10], from which it will be clear how they combine to yield
Theorem 1 above. Theorem 31.1 [10] is a rather technical result which,
very roughly speaking, asserts that if $f$ is any partial recursive regressing
function, then there is an r.e. set $C$ and a partial recursive retracing func-
tion $p$, such that if $A$ is (the set of nodes of) an infinite branch of the graph
of $f$, then $C \cap A$ is an infinite retraceable set retraced by $p$ and, moreover,$C \cap A$ when arranged in natural order has very strong order-preservation
properties with respect to the class of partial recursive functions. (Actually,
Theorem 31.1 of [10] asserts more; we have indicated only the portion
we needed.) Theorem 31.2 [10] asserts that if the infinite branches of a
partial recursive retracing function have the order-preservation properties
of [10, Theorem 31.1], then they are all "thin" in the sense of enjoying
both property (P) and property (Q). Theorem 1 of the present paper
follows, since to be regressive is precisely to be (the set of nodes of) a
branch of a partial recursive regressing function.

In his interesting recent paper [1], A. N. Degtev has proved the following
theorem (among others):

**Theorem 2 (Degtev).** Suppose $A$ is an infinite retraceable set such
that $\bar{A}$ is r.e., and such that $A$ has property (P). Then if $B$ is any infinite
cô-r.e. subset of $A$, $B$ is retraceable.

We here observe that a somewhat stronger form of Theorem 2 holds,
namely:

**Theorem 2'.** If $A$ is any infinite retraceable set having property (P),
and if $B$ is any co-r.e. set such that $A \cap B$ is infinite, then $A \cap B$ is retraceable.

Though the proof of Theorem 2' is not difficult, the theorem itself was
overlooked by the author of the present note in his fairly extensive investi-
gation [10] of sets with property (P). As an adequate indication of the
proof, we offer the following: Since $A$ has property (P), the principal
function $p_A$ of $A$ satisfies the condition

$$(\exists m)(\forall n)[(n > m \& g(p_A(n)) \text{ defined}) \Rightarrow p_A(n + 1) > g(p_A(n))]$$

where $g(n) = \delta_f(\mu y)$ [y is the Gödel number of a computation of $\varphi_e(n)$] with
$e$ chosen so that $\bar{B} =$ the domain of $\varphi_e$. This fact allows us to tell of a
number $p_A(n + 1)$ whether $p_A(n)$ is in $\bar{B}$, with finitely many exceptions
(which of course do not matter).²

We come now to our main assertion and its corollary. In the statement

² It is easily seen from this proof that a further strengthening of Theorem 2 is possible;
namely, in Theorem 2' we need not require that $B$ be co-r.e. but only that it lie in the
boolean algebra generated by the r.e. sets.
of the corollary, highly dense-simple means r.e. with complement having property (Q); while point-decomposability is to be understood as defined in [8]. (The notion of (not necessarily high) dense simplicity was first introduced in [7].)

**Theorem 3.** Let $A$ be an infinite regressive set. Then there exists a recursively enumerable set $B$ such that

1. $A \cap \overline{B}$ is infinite and retraceable and has properties (P) and (Q),
2. $(\forall C)((C \text{ r.e.} \& B \subseteq C \cap A \cap \overline{C} \text{ infinite}) \Rightarrow A \cap \overline{C} \text{ is retraceable}).$

**Proof.** Applying Theorem 1 to $A$, we obtain an r.e. set $B$ such that $A \cap \overline{B}$ is infinite, retraceable, and has properties (P) and (Q). By property (P) and Theorem 2', $A \cap \overline{C}$ is retraceable for any r.e. set $C$ satisfying $\exists \overline{C} = C \cap A \cap \overline{C}$ infinite, and we are done.

**Remark.** It is easily shown that property (P) is hereditary for retraceable subsets; hence, in the statement of Theorem 3, we can strengthen (ii) by asserting that $A \cap \overline{C}$ has property (P) as well as being retraceable.

As remarked in [1], if $A$ is r.e. and coinfinite and can be extended to an r.e. superset $B$ such that $B$ has a point-decomposable complement, then $A$ can be extended to an r.e. set $C$ such that $\overline{C}$ is infinite, immune, and regressive. We therefore obtain the following corollary to Theorem 3, which provides yet another refinement (see Theorems in [7], [6], and [11]) of Martin's result that hypersimple sets need not have maximal supersets:

**Corollary 1.** Let $A$ be any r.e. set which can be extended to an r.e. set $B$ having a point-decomposable complement. Then $A$ can be extended to a highly dense-simple set $C$ all of whose co-infinite r.e. extensions are co-retraceable.

**Proof.** Property (Q), for the complement of an r.e. set, is precisely our notion of high dense simplicity.

**Remark.** A weaker version of Corollary 1 is certainly already present in [1], since Degtev there exhibits his own construction of a particular co-r.e. retraceable set having property (P). The latter construction can in fact be modified to take place inside a given infinite retraceable set with r.e. complement, although this is not done in [1]; such a modification leads at once to another proof of Theorem 1 for the special case in which $A$ is r.e.

2. **A further application of Theorem 1, and a concluding remark relating [1] and [5].** C. G. Jockusch has proved that no r.e. set can be both dense simple (in the sense of [7]) and strongly effectively simple. (See [5] for the meaning of strong effective simplicity; the standard example is the original simple-but-not-hypersimple set constructed by E. L. Post.)
From Jockusch's result and Theorem 1, since strong effective immunity is trivially hereditary, we have:

**Corollary 2.** The complement of an infinite, co-r.e. regressive set cannot be strongly effectively simple. (It is not hard to show, on the other hand, that such a set can be merely effectively simple; again, see [5] for the definition of effective simplicity. We are indebted to Jockusch for pointing out Corollary 2.)

**Proof.** By Theorem 1 we have that each infinite, co-r.e. regressive set can be shrunk to an infinite, co-r.e. retraceable set having property (Q) and hence having a highly dense-simple complement. Now apply Jockusch's theorem on the incompatibility of dense simplicity and strong effective simplicity, noticing that any highly dense-simple set is certainly dense simple in the sense of [7].

Jockusch has suggested that we remark also on the fact that Degtev has shown, in [1], that every semirecursive regressive set is either r.e. or co-r.e. This not only answers a question raised in [5], but, in light of Corollary 2 above, it shows that Theorem 6.4 of [5] is vacuous.

We would like to emphasize, in conclusion, that the really crucial observation for this note is Degtev’s simple but rather striking Theorem 2.

**References**


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