CRITERIA FOR COMPACTNESS AND FOR DISCRETENESS OF LOCALLY COMPACT AMENABLE GROUPS

EDMOND GRANIRER

Abstract. Let $G$ be a locally compact group $P(G) = \{0 \leq \phi \in L_1(G); \int \phi(x) \, dx = 1\}$ and $(L_\phi)(x) = \phi(x) = f(ax)$ for all $a, x \in G$ and $f \in L^\infty(G)$. $0 \leq \Psi \in L^\infty(G)\ast$, $\Psi(1) = 1$ is said to be a topological left invariant mean (TLIM LIM) if $\Psi(af) = \Psi(f)$ for all $a \in G, \phi \in P(G), f \in L^\infty(G)$. The main result of this paper is the

Theorem. Let $G$ be a locally compact group, amenable as a discrete group. If $G$ contains an open $\sigma$-compact normal subgroup, then $\text{LIM} = \text{TLIM}$ if and only if $G$ is discrete. In particular if $G$ is an infinite compact amenable as discrete group then there exists some $\Psi \in \text{LIM}$ which is different from normalized Haar measure. A harmonic analysis type interpretation of this and related results are given at the end of this paper.

Introduction. It was known to Fred Greenleaf that if $T$ is the circle group then there are at least two different linear translation invariant functionals $\Psi \geq 0$ on $L^\infty(T)$ with $\Psi(1) = 1$. One of them is certainly that given by the normalized Haar measure $\lambda$ on $T$.

It is easy to show and it is known that on any compact $G$, $\lambda$ is the unique $0 \leq \Psi \in L^\infty(G)\ast$, $\Psi(1) = 1$ which satisfies the stronger invariance property $\Psi(\phi \ast f) = \Psi(f)$ for all $f \in L^\infty(G), \phi \in P(G)$ (i.e. $\lambda$ is the unique TLIM on $L^\infty(G)$). This is the case since $\phi \ast f \in C(G)$ for all $\phi \in P(G), f \in L^\infty(G)$ and if $\Psi \in \text{TLIM}$ then $\Psi \in \text{LIM}$ [6, p. 25]. Thus $\Psi = \lambda$ at least on $C(G)$. But then for all $f \in L^\infty(G), \Psi(f) = \Psi(\phi \ast f) = \lambda(\phi \ast f) = \lambda(f)$.

It seemed to Greenleaf that for any compact infinite $G$, which is amenable as a discrete group, there exist at least two different LIM’s on $L^\infty(G)$. Our main result in this paper implies the

Theorem. Let $G$ be a locally compact group which is abelian or $\sigma$-compact and amenable as a discrete group. Then $\text{LIM} = \text{TLIM}$ if and
only if \( G \) is discrete. In particular on any compact infinite \( G \) which is amenable as discrete there exists some \( \Psi \in \text{LIM} \) different from the normalized Haar measure.

Let \( H \) \( [H_c] \) be the linear span of \( \{ f - l_a f; f \in L^\infty(G), \ a \in G \} \) \( \{ [f - \phi * f]; \ \phi \in P(G), \ f \in L^\infty(G) \} \) and for \( A \subset L^\infty(G) \) let \( \bar{A} \ [A^*] \) denote the norm \( [w^*] \) closure of \( A \) in \( L^\infty(G) \). In any locally compact group one has \( \bar{H} \subset H_c \subset \bar{H}_c^* = \bar{H}^* = L^\infty(G) \). Our last result (combined with some known facts) when restricted to \( \sigma \)-compact locally compact abelian groups runs as follows:

**Proposition.**  (i) If \( G \) is compact and infinite then \( \bar{H} \subseteq \bar{H}_c = \bar{H}_c^* = \bar{H}^* = \{ f \in L^\infty(G); \lambda f = 0 \} \).

(ii) If \( G \) is not compact then \( \bar{H} \subset \bar{H}_c \subset \bar{H}_c^* = \bar{H}^* = L^\infty(G) \). Moreover \( L^\infty(G) \bar{H}_c \) is a nonseparable Banach space and \( \bar{H} = \bar{H}_c \) iff \( G \) is discrete.

We conjecture at the end that for any locally compact amenable group \( G \), if \( G \) is noncompact then \( L^\infty(G) \bar{H}_c \) is a nonseparable Banach space and if \( G \) is nondiscrete then \( \bar{H}_c / \bar{H} \) is nonseparable (with induced quotient norms).

**Some more notations.** Unless otherwise specified we assume the notations and definitions of Hewitt-Ross [7].

If \( G \) is a locally compact group \( \lambda \) will denote a fixed left Haar measure (with \( \lambda(G) = 1 \) if \( G \) is compact), we write sometimes \( \int \phi(x) \, dx \) instead of \( \int \phi \, d\lambda \).

\( \Psi \in L^\infty(G)^* \) is said to be [topologically] left invariant if \( \Psi(l_a f) = \Psi(f) \) \( [\Psi(\phi * f) = \Psi(f)] \) for all \( f \in L^\infty(G), \ \phi \in P(G), \ a \in G \) (where \( l_a f(x) = a f(x) = f(ax) \)). If \( \Psi \) satisfies in addition \( \Psi \geq 0 \) and \( \Psi(1) = 1 \) then \( \Psi \) is said to be a [topological] left invariant mean ([TLIM] LIM resp.). The set of all [TLIM] LIM is also denoted by [TLIM] LIM. Analogously we define [TRIM] RIM the sets of [topological] right invariant means.

We stress that LIM, TLIM are both included in \( L^\infty(G)^* \). The locally compact group \( G \) is said to be amenable if \( \text{LIM} \neq \emptyset \) (or equivalently if \( \text{TLIM} \neq \emptyset \) see [6]). \( G \) is said to be amenable as discrete if \( G_d \) (i.e. \( G \) with the discrete topology) is amenable.

We write sometimes \( \text{LIM}(G) \), \( \text{TLIM}(G) \) to emphasize dependence on the group \( G \). If \( A \subset G \), \( 1_A \) denotes the function 1 on \( A \) and zero otherwise. If \( \Psi \in L^\infty(G)^* \), we write \( \Psi(B) \) instead of \( \Psi(1_B) \) for measurable \( B \subset G \). 1 also stands for the constant one function on \( G \).

**Proposition 1.** Let \( G \) be any noncompact locally compact group and \( \phi \in \text{TRIM} \). If \( B \) is a measurable set and \( \lambda(B) < \infty \) then \( \phi(B) = 0 \).
Proof. Let $\phi_x \in P(G)$ be such that $\phi_x \rightarrow \phi$ in $w^*$ and let $\eta \in P(G)$ be such that $0 \leq \eta(x) \leq \varepsilon$ for all $x \in G$. Then

$$|\phi_x * \eta(x)| = \left| \int \phi_x(y) \eta(y^{-1}x) \, d\lambda \right| \leq \varepsilon \int \phi_x(y) \, d\lambda = \varepsilon.$$ 

Furthermore if $f \in L^\infty(G)$ then

$$(\phi_x * \eta)(f) = \phi_x(f * \eta) \rightarrow \phi(f * \eta) = \phi(f).$$

(See Wong [10, p. 352].) Hence if $f \in L^\infty \cap L^1$ then $|\phi_x * \eta(f)| \leq \int |f| \, d\lambda$ so $|\phi f| \leq (\int |f| \, d\lambda) \varepsilon$. Thus $\phi f = 0$.

We need the following, probably known, proposition for which we were unable to find a reference.

**Proposition 2.** Let $G$ be a $\sigma$-compact nondiscrete locally compact group. Then for any $\varepsilon > 0$ there exists an open dense set $B \subset G$ with $\lambda(B) < \varepsilon$.

Proof. It is enough to show the existence of a dense set $D \subset G$ with $\lambda(D) = 0$ and the regularity of $\lambda$ would imply that for some open $D \subset B$, $\lambda(B) < \varepsilon$.

If $G$ is separable then there is some countable dense $D \subset G$. Clearly $\lambda(D) = 0$.

Assume now that $G$ is arbitrary and $N \subset G$ a closed normal subgroup. Let $\theta : G \rightarrow G/N$ be the canonical map. If $D \subset G$ with $\theta D$ dense in $G/N$ then $DN$ is dense in $G$. In fact if $U \subset G$ is open with $U \cap DN = \emptyset$ then $UN \cap DN = \emptyset$ so $\theta^{-1}(\theta U \cap \theta D) = UN \cap DN = \emptyset$ thus $\theta U \cap \theta D = \emptyset$ and $\theta U$ is open in $G/N$ which cannot be.

If $G$ is $\sigma$-compact nondiscrete let $U \subset G$ be an open neighborhood of the identity and let $G_0 = \bigcup_{n=0}^{\infty} U^n$. Then $G_0$ is open compactly generated and there are countably many left cosets of $G$ w.r.t. $G_0$. The left Haar measure of $G_0$ can be taken to be the restriction to $G_0$ of the left Haar measure $\lambda$ on $G$. It is enough hence to show that there is a dense null set $D \subset G_0$ i.e. we can and shall assume that $G$ is compactly generated nondiscrete. Let then $U_n$ be a sequence of identity neighborhoods in $G$ with $\lambda(U_n) \rightarrow 0$ and let $N \subset \bigcap_{n=0}^{\infty} U_n$ be a compact normal subgroup such that $G/N$ is metrizable separable (see [7, p. 71]). ($G/N$ is not discrete since $\lambda N = 0$ so $N$ is not open.) Let $D = \{d_i\}^{\infty}_{i=0} \subset G$ be such that its image in $G/N$ is dense. Then $DN \subset G$ is dense and $\lambda(DN) = 0$ since $D$ is countable.

We need in the sequel the following proposition (not in its full force) which is in part due to Følner [3] for discrete amenable groups.

**Proposition 3.** Let $G$ be a locally compact group which is amenable as a discrete group. For $f \in L^\infty(G)$ let $M(f) = \sup\{\phi(f); \phi \in LIM\}$. Then
for all $f \in L^\infty(G)$

$$Mf = \inf_{\mathcal{A}} \esssup_x \left[ \frac{1}{n} \sum_{i=1}^{n} f(a_i x) \right]$$

the inf being taken over the set $\mathcal{A}$ of all finite tuples $(a_1, \cdots, a_n)$ of elements of $G$.

**Proof.** Let $H$ be the linear span of $\{f - l_a f; \ a \in G, f \in L^\infty(G)\}$. It is known (and due to Følner [3, p. 6] for discrete $G$) that:

$$M(f) = \inf_{h \in H} \esssup_x (f(x) + h(x))$$

for all $f \in L^\infty(G)$. (For an easy proof see [5, p. 401].)

Also if $\phi \in \text{LIM}$ then $\phi f = \phi(n^{-1} \sum l_i f)$ hence

$$Mf \leq \inf_{\mathcal{A}} \esssup_x \frac{1}{n} \sum_{i=1}^{n} f(a_i x).$$

Let now $\varepsilon > 0$ and $h_0 \in H$ be such that $M(f) + \varepsilon > \esssup_x (f(x) + h_0(x))$. So $M(f) + \varepsilon \geq f(x) + h_0(x)$ locally a.e. and a fortiori $M(f) + \varepsilon \geq \frac{1}{n} \sum l_i f(x) + h_0(x)$ loc. a.e. for all $a_1, \cdots, a_n$ in $G$. We claim that a finite set $\{b_1, \cdots, b_k\} \subseteq G$ can be chosen such that $|k^{-1} \sum l_b f(x)| < \varepsilon/2$ loc. a.e. This would imply that $M(f) + 3/2 \varepsilon < k^{-1} \sum l_b f(x)$ loc. a.e., i.e. that

$$M(f) \geq \inf_{\mathcal{A}} \esssup_x \frac{1}{n} \sum_{i=1}^{n} l_i f(x)$$

which would end this proof.

To prove this claim let $h_0 = \sum [f_i - l_i f_i]$. For the finite set $F = \{c_1, \cdots, c_n\}$ choose a finite subset $A = \{b_1, \cdots, b_k\}$ to satisfy $c(A) < \delta c(A)$ for $1 \leq i \leq n$ where $c(B)$ stands for the cardinality of $B$ and $\delta = \varepsilon (\max_{1 \leq i \leq n} \|f_i\|)^{-1} n^{-1}$. Such $A$ can be found by Følner’s characterization of discrete amenable groups [2] (see Namioka [9, p. 22]). Then for each $i \leq n$

$$\left| \frac{1}{k} \sum_{j=1}^{k} (c_i b_j - l_j f_i) \right| = \left| \frac{1}{k} \sum_{j=1}^{k} (l_j b_j - l_j f_i) \right| \leq c(c_i A - A) \|f_i\|/c(A) < \delta \|f_i\| \leq \varepsilon/n.$$

Therefore $|k^{-1} \sum l_b h_0(x)| \leq \varepsilon/2$ loc. a.e. which finishes this proof.

**Remarks.** 1. It seems that this proposition does not hold true if $G$ is not amenable as a discrete group (even in the case that $G$ is compact).

2. If $m(f) = \inf \{\langle f, \phi \rangle; \ \phi \in \text{LIM}\}$ then

$$m(f) = -M(-f) = \sup_{\mathcal{A}} \left[ \essinf_x \frac{1}{n} \sum_{i=1}^{n} f(a_i x) \right].$$
3. One can show in a similar way that the support functional of the set of two-sided invariant means is $M_0f = \inf_{\mathcal{A}} \text{ess sup} \frac{1}{nm} \sum_{i,j} f(a_i b_j)$ where $\mathcal{A}$ is the set of all pairs of finite tuples $(a_1, \cdots, a_n)(b_1, \cdots, b_m)$ of elements of $G$.

**Theorem 1.** Let $G$ be a locally compact $\sigma$-compact group which is amenable as a discrete group. If $\text{LIM} = \text{TLIM}$ then $G$ is discrete.

**Proof.** Assume that $G$ is not discrete and let $O$ be an open dense set in $G$ with $\lambda(O) < \frac{1}{2}$. Thus, if $G$ is not compact then $\phi(O) = 0$ for all $\phi \in \text{TRIM}$ hence $\Psi(O^{-1}) = 0$ for all $\Psi \in \text{TLIM}$ (see [4, p. 50]). If $G$ is compact then $\lambda(O^{-1}) = \lambda(O) < \frac{1}{2}$. Let $B = G \sim O^{-1}$. Then $B$ is closed nowhere dense, $\Psi(B) = 1$ if $\Psi \in \text{TLIM}$ and $G$ is not compact while $\lambda(B) > \frac{1}{2}$ if $G$ is compact. (In this last case $\{\lambda\} = \text{TLIM} = \text{TRIM}$.)

In different terminology $B$ is topologically almost left convergent to 1 (or to a positive real $> \frac{1}{2}$ if $G$ is compact).

We claim that $\phi(B) = 0$ for some $\phi \in \text{LIM}$. If not, then

$$m(1_B) = \inf_{\phi \in \text{LIM}} \text{ess sup} \frac{1}{n} \sum_{i=1}^{n} \phi(a_i x) = d > 0.$$  

But then there are $b_1, \cdots, b_k$ in $G$ such that $\text{ess sup}_x k^{-1} \sum_{i=1}^{k} \lambda_1 B(b_i x) \geq d/2$ i.e. locally a.e. in $x$ one has $k^{-1} \sum_{i=1}^{k} \lambda_1 B(x) \geq d/2 > 0$. But this contradicts the fact that $A = G \sim \bigcup_{i=1}^{k} B_i^{-1} B$ is open dense, hence of nonzero Haar measure and for $x \in A$, $k^{-1} \sum_{i=1}^{k} \lambda_1 B(b_j x) = 0$. Using Remark 3 above one could easily show that in fact $\phi(B) = 0$ for some two sided invariant mean $\phi$ on $L^\infty(G)$.

**Remarks.** Let $G$ be a locally compact amenable group with $G_0 \subseteq G$ an open subgroup. Let $\lambda$, $\lambda_0$ be the Haar measures on $G$, $G_0$. As known and easily shown the $\lambda_0$ measurable sets comprise exactly the $\lambda$ measurable sets of $G$ which are included in $G_0$. We can and shall choose $\lambda_0$ to be the restriction of $\lambda$ to $G_0$. (We use the terminology of [7].)

For $f \in L^\infty(G)$ define $(\pi f)(x) = f(x)$ for $x \in G_0$. Then $\pi$ can be considered as a map onto $L^\infty(G_0)$. If $v \in L^\infty(G_0)^*$ is left invariant and $f \in L^\infty(G)$, let $(S_v f)(z) = v(\pi l_z f)$ for all $z \in G$. Let $\{z_i G_0\}_{i \in I}$ be a fixed decomposition of $G$ into left cosets w.r.t. $G_0$. Then, the bounded function, $S_v f$ is constant on each $z_i G_0$ (as known) since if $z = z_{i} a$, $a \in G_0$ then $S_v f(z a) = v(\pi l_z a f) = v(\pi l_z f) = S_v f(z)$, since $a \in G_0$. This implies that $S_v f \in UCB_t(G)$ (i.e. is left uniformly continuous as in [7] for all $f \in L^\infty(G)$ and left invariant $v \in L^\infty(G_0)^*$). This is the case since for all $z \in G$, $x \in G_0$, $S_v f(z x) - S_v f(z) = 0$ and $G_0$ is open.

Choose and fix now some LIM, $\mu_0$ on $C(G)$ and define for any left invariant $v \in L^\infty(G_0)^*$, $Tv \in L^\infty(G)^*$, by $Tv(f) = \mu_0(S_v f)$. 

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As known and readily checked $T$ maps the set of left invariant elements $[\text{LIM}]$ of $L^\infty(G_0)^*$ into the set of left invariant elements $[\text{LIM}]$ of $L^\infty(G)^*$.

The above is a refinement of a construction due to M. M. Day [1, p. 533]. In the above context we have the

**Proposition 4.** Let $G$ be a locally compact amenable group and $G_0 \subseteq G$ an open normal subgroup.

If $Tv \in \text{TLIM}(G)$ for some $v \in \text{LIM}(G_0)$ then $v \in \text{TLIM}(G_0)$.

**Proof.** If $f \in L^\infty(G_0)$ denote by $f_1$ its $\{z_\alpha\}$ periodic extension i.e. $f_1(z_\alpha x) = f(x)$ for all $x \in G_0$ and all $\alpha$. (Note that $\{z_\alpha\}$ are fixed.) It is clear that $f_1$ is measurable (since it needs to be so only on compacta [7, p. 131], and $G_0$ is open).

If $z \in z_\alpha G_0$ then

\[ S_v(f_1)(z) = S_v(f)(z) = \nu(\pi l_{z_\alpha} f_1) = \nu(f) \]

since if $x \in G_0$ then $(\pi l_{z_\alpha} f_1)(x) = f_1(z_\alpha x) = f(x)$. Thus $T_v f_1 = \mu_0(S_v f_1) = \mu_0(\nu f \cdot 1_G) = \nu f$.

Fix now $\phi_0 \in P(G)$ with support included in $G_0$. Then for $f \in L^\infty(G_0)$ and $x \in G_0$ one has:

\[ l_{z_\alpha}(\phi_0 * f_1)(x) = \int f_1(y^{-1} z_\alpha x) \phi_0(y) dy = \int f_1((z_\alpha y z_\alpha^{-1})^{-1} z_\alpha x) \phi_0(z_\alpha y z_\alpha^{-1}) \Delta(z_\alpha^{-1}) dy = \int f_1(z_\alpha y^{-1} x) \phi_0(z_\alpha y z_\alpha^{-1}) \Delta(z_\alpha^{-1}) dy = \int f(y^{-1} x) \phi_0(y z_\alpha y z_\alpha^{-1}) \Delta(z_\alpha^{-1}) dy = (\Psi_\alpha \otimes f)(x) \]

where $\Psi_\alpha(y) = \phi_0(z_\alpha y z_\alpha^{-1}) \Delta(z_\alpha^{-1})$ for $y \in G_0$, thus $\Psi_\alpha \in P(G_0)$ and where $\otimes$ stands for convolution in $L_1(G_0)$. Note, that since $G_0$ is normal $\phi_0(z_\alpha y z_\alpha^{-1})$ has support included in $G_0$.

It follows that if $z \in z_\alpha G_0$ then

\[ S_v(\phi_0 * f_1)(z) = S_v(\phi_0 * f_1)(z) = \nu(\pi l_{z_\alpha} (\phi_0 * f_1)) = \nu(\phi_0 \otimes f). \]

Note that we have used in the last equality only the fact that $v \in \text{LIM}(G_0)$.

From it alone, it follows (see Greenleaf [6, proof of Lemma 222, p. 27]) that $\nu(\phi \otimes f) = \nu(\Psi \otimes f)$ for all $\phi, \Psi \in P(G_0)$.

Hence $T_v(\phi_0 * f_1) = \mu_0(S_v(\phi_0 * f_1)) = \nu(\phi_0 \otimes f)$. 

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But by assumption $Tv \in \text{TLIM}$. Thus $Tv(\phi_0 \ast f_1) = (Tv)f_1 = \nu f$ and hence, for all $f \in L^\infty(G_0)$, $\nu(\phi_0 \otimes f) = \nu(f)$. The above remark implies that $\nu \in \text{TLIM}(G_0)$ and finishes this proof.

**Theorem 2.** Let $G$ be a locally compact group which is amenable as a discrete group. Assume that $G$ contains a $\sigma$-compact open normal subgroup. If $\text{LIM}(G) = \text{TLIM}(G)$ then $G$ is discrete.

**Remark.** 1. If $G$ has equivalent left and right uniform structures then $G$ contains a neighborhood $U$ of the identity with compact closure such that $xUx^{-1} = U$ for all $x \in G$. Thus $G_0 = \bigcup_{n=0}^{\infty} U^n$ is normal $\sigma$-compact and open. In particular the theorem certainly holds true for all locally compact abelian groups $G$. It also holds true for all $\sigma$-compact $G$ which are amenable as discrete groups (take $G = G_0$).

2. We could have assumed in this theorem that $G$ is a locally compact amenable group and the open normal $\sigma$-compact $G_0$ is amenable as discrete. This however readily implies that $G$ is amenable as discrete and we would not gain anything. (The discrete $G/G_0$ and $G_0$ with discrete topology are amenable hence so is $G$ with discrete topology.)

**Proof.** If $\text{TLIM}(G) = \text{LIM}(G)$ then $\text{TLIM}(G_0) = \text{LIM}(G_0)$ since $Tv \in \text{TLIM}(G) = \text{LIM}(G)$ for all $\nu \in \text{LIM}(G_0)$. Thus $\nu \in \text{TLIM}(G_0)$ by the previous proposition. We use now Theorem 1 and get that $G_0$ is discrete. Thus if $x \in G_0$, $\{x\}$ is open in $G_0$ hence in $G$. Hence $G$ is discrete.

The following is an interpretation of our and some known related results from the point of view of harmonic analysis on locally compact groups.

Let $H$ [Hc] denote the linear span of $\{f - l_x f; f \in L^\infty(G), x \in G\}$ or $\{f \ast \phi f; f \in L^\infty(G), \phi \in P(G)\}$. If $A \subset L^\infty(G)$ denote by $\tilde{A}$ [$\tilde{A}^*$] its norm [$w*$] closure.

We need the following known remark whose proof uses a trick due to I. Namioka [9].

**Remark.** Let $\Psi$, $\Psi_1$, $\Psi_2 \in L^\infty(G)^*$, $\phi \in P(G)$ and define $(L_\phi \Psi)f = \Psi(\phi \ast f)$ for $f \in L^\infty(G)$. Let $\Psi_1 \vee \Psi_2 = \max(\Psi_1, \Psi_2)$ in the lattice $L^\infty(G)^*$ and $\Psi^+ = \Psi \vee O$, $\Psi^- = (-\Psi) \vee O$. If $\Psi \in L^\infty(G)^*$ satisfies $L_\phi \Psi = \Psi$ for all $\phi \in P(G)$, then so do $\Psi^+$ and $\Psi^-$. If $\phi \in P(G)$, $L_\phi (\Psi \vee O) \geq (L_\phi \Psi \vee L_\phi O) = \Psi \vee O = \Psi^+$. So $L_\phi \Psi^+ = \Psi^+ \geq 0$. But $(L_\phi \Psi^+ - \Psi^+)(1) = 0$. Thus $L_\phi \Psi^+ = \Psi^+$. (Same true, if $L_\phi$ is replaced by $L_\phi^*$ for all $a \in G$.)

**Proposition 5.** (a) Let $G$ be compact and infinite. Then

$$\tilde{A} \subset \tilde{A}_c = \tilde{A}_c^* = \tilde{A}^* = \{f \in L^\infty(G); \lambda f = 0\}.$$ 

If $G$ is abelian (or even amenable as a discrete group) then $\tilde{A} \neq \tilde{A}_c$.
(b) Let $G$ be a noncompact locally compact group. Then $\tilde{H} \subset \tilde{H}_c \subset H^* = H_c^* = L^\infty(G)$. Furthermore

(i) $\tilde{H}_c \subset L^\infty(G)$ iff $\tilde{H} = L^\infty(G)$ iff $G$ is not amenable (i.e. $\text{LIM} = \emptyset$).

(ii) If $G$ is $\sigma$-compact amenable then $L^\infty(G)/\tilde{H}_c$ is a nonseparable Banach space.

(iii) If $G$ is a $\sigma$-compact and amenable as discrete or amenable and containing such an open normal subgroup (in particular if $G$ is locally compact abelian), then $\tilde{H} = \tilde{H}_c$ iff $G$ is discrete.

Proof. (a) $\tilde{H} \subset \tilde{H}_c$ is due to the fact that $\text{TLIM} \subset \text{LIM}$ [6, p. 25], the remark above and the Hahn-Banach theorem (this part with $G$ not necessarily compact). Thus $H^* \subset \tilde{H}^*$. If the inclusion were proper then there would exist some $\phi \in L_1(G)$ such that $\phi(H) = 0$ but $\phi(g) \neq 0$ for some $g \in H_c$. But then $\phi$ is left invariant and in $L_1(G)$ hence $\phi = c\lambda$ for some scalar $c \neq 0$. Hence $\phi(H_c) = \lambda(H_c) = 0$ which cannot be. So $\tilde{H} \subset \tilde{H}_c \subset H^* = H_c^* = \{f \in L^\infty(G); \lambda f = 0\}$.

That $\tilde{H}_c = \{f \in L^\infty(G); \lambda f = 0\}$ is a consequence of Theorem 7.3, p. 360 of J. C. S. Wong [10] or can directly be proven. The rest of (a) is implied by the main theorem of this paper.

(b) If $H^* \neq L^\infty(G)$ there would exist $0 \neq \phi \in L_1(G)$ such that $\phi(H) = 0$. But then $\phi$ is left invariant hence so are $\phi^+$, $\phi^-$ and $\phi^+ \neq 0$ or $\phi^- \neq 0$. Assuming that $\phi^+ \neq 0$, $\mu(A) = \int_A \phi^+ d\lambda$ is a measure on the Borel sets of $G$ satisfying all the conditions in Hewitt-Ross [7, p. 194]. Hence $\mu = c\lambda$ for some $c > 0$ (since $\mu \neq 0$).

Since $\mu(G) < \infty$, $\lambda(G) < \infty$ so $G$ is compact. That (b)(i) holds is known and readily shown. (b)(ii) is shown as follows: If $L^\infty(G)/\tilde{H}_c$ would be separable there would exist a sequence $\{f_n\} \subset L^\infty(G)$ such that (if $B$ is the linear span of $\{f_n\}$) $\tilde{H}_c + B$ is norm dense in $L^\infty(G)$ (see [4, p. 63]). But $\tilde{H}_c = \{f \in L^\infty(G); \forall \Psi \in \text{TLIM}; \Psi(f) = 0\}$ for all $\Psi \in \text{TLIM}$ Wong [10, p. 360]. Fix now some $\Psi_0 \in \text{TLIM}$ and let $\Psi_0 f_n = \alpha_n$. Then $\{\Psi_0 f_n = \alpha_n\}$ since any $\Psi$ which belongs to the right side will coincide with $\Psi_0 \geq 1$ on $\tilde{H}_c + B$ hence on $L^\infty(G)$. We apply now [4, Theorem 5, p. 53] with $K = P(G)$ hence $A = \{\Psi \in \text{TLIM}; \forall \Psi_0 f_n = \alpha_n = 0\}$ is norm separable. Thus $G$ is compact. (b)(iii) is just our main theorem and the fact that $\tilde{H} = \tilde{H}_c$ iff $\text{LIM} = \text{TLIM}$ (by our remark above and the Hahn-Banach theorem).

Main conjecture. Let $G$ be any amenable locally compact group. If $G$ is noncompact then $L^\infty(G)/\tilde{H}_c$ is nonseparable. If $G$ is nondiscrete then $\tilde{H}_c/\tilde{H}$ is nonseparable.

Addition. In the meantime W. Rudin sent us a preprint of a paper of his, in which he proves Theorem 2 without the assumption that (*) "$G$ contains an open $\sigma$-compact normal subgroup", but with the assumption
that $G$ is amenable as discrete. His proof is different from ours and uses harmonic analysis type arguments. After reading his manuscript we found the following easy argument which removes the restriction (*).

**Proposition.** Let $G_0$ be an open noncompact subgroup of $G$, and

$$G = \bigcup_{\alpha \in \Lambda} x_\alpha G_0, \quad x_\alpha G_0 \cap x_\beta G_0 = \emptyset \quad \text{if } x \neq x_\beta.$$  

If $A_0 \subseteq G_0$ is such that $\lambda(A_0) < \infty$ (where $\lambda$ is the Haar measure on $G$) then for all $\phi \in \text{TRIM}$, $\phi(\bigcup_{\alpha \in \Lambda} x_\alpha A_0) = 0$.

**Proof.** Let $B_n \subseteq G_0$ be compact with $\lambda(B_n) = a_n \uparrow \infty$ and let $f_n = a_n^{-1} 1_{B_n}$, $A = \bigcup_{\alpha} x_\alpha A_0$. Then

$$1_A * f_n^\sim(x) = a_n^{-1} \int 1_A(y) 1_{B_n}(y^{-1} x) \, dy$$

$$= a_n^{-1} \lambda(x B_n \cap A) \leq a_n^{-1} \lambda(x G_0 \cap A)$$

$$= a_n^{-1} \lambda(x_\alpha G_0 \cap A) = a_n^{-1} \lambda(x_\alpha A_0) = a_n^{-1} \lambda(A_0),$$

for some (hence all) $\alpha \in \Lambda$.

If $\phi \in \text{TRIM}$ then $\phi(A) = \phi(1_A * f_n^\sim) \leq a_n^{-1} \lambda(A_0) \to 0$.

**Remark.** If $\Psi \in \text{TLIM}$, then $\Psi(\bigcup_{\alpha} A_0^{-1} x_\alpha^{-1}) = 0$. (See [4, pp. 49-50].)

To remove restriction (*) on $G$, let $G_0$ be any $\sigma$-compact, noncompact, open subgroup of $G$, if $G$ is noncompact, and $G = G_0$, if $G$ is compact. Let $A_0 \subseteq G_0$ be open dense with $\lambda(A_0) \leq \frac{1}{2}$ and $A = \bigcup_{\alpha} A_0^{-1} x_\alpha^{-1}$ (as above), $A = A_0$ if $G$ is compact. Let $B = G \sim A$. Then $\psi(B) = 1$ for all $\psi \in \text{TLIM}$, if $G$ is not compact, $\lambda(B) \geq \frac{1}{2}$ if $G$ is compact. $B$ is closed nowhere dense. Continue now as in the proof of Theorem 1.

**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, BRITISH COLUMBIA, CANADA