ON AKCOGLU AND SUCHESTON'S OPERATOR
CONVERGENCE THEOREM
IN LEBESGUE SPACE

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Abstract. Let \( T \) be a bounded linear operator on an \( L_1 \)-space
and \( \tau \) its linear modulus. It is proved that if the adjoint of \( \tau \) has
a strictly positive subinvariant function then the following two
conditions are equivalent: (i) \( T^n \) converges weakly; (ii) \( (1/n) \sum_{k=1}^{n} T^k \)
converges strongly for any strictly increasing sequence \( k_1, k_2, \ldots \) of
nonnegative integers.

1. Introduction. Let \( (X, \mathcal{M}, m) \) be a \( \sigma \)-finite measure space and
\( L_p(X) = L_p(X, \mathcal{M}, m), 1 \leq p \leq \infty \), the usual (complex) Banach spaces. If
\( A \in \mathcal{M} \) then \( 1_A \) is the indicator function of \( A \) and \( L_p(A) \) denotes the Banach
space of all \( L_p(X) \)-functions that vanish a.e. on \( X - A \). Let \( T \) be a bounded
linear operator on \( L_1(X) \) and \( \tau \) its linear modulus [2]. Thus \( \tau \) is a positive
linear operator on \( L_1(X) \) such that

\[
\|\tau\|_1 = \|T\|_1 \quad \text{and} \quad \tau g = \sup\{|Tf|; f \in L_1(X) \text{ and } |f| \leq g\}
\]

for any \( 0 \leq g \in L_1(X) \). The adjoint of \( T \) is denoted by \( T^* \). Clearly \( T \) is a
contraction if and only if \( \tau 1 \leq 1 \). In [1] Akcoglu and Sucheston proved
that if \( T \) is a contraction then the following two conditions are equivalent:
(i) \( T^n \) converges weakly; (ii) \( (1/n) \sum_{k=1}^{n} T^k \) converges strongly for any
strictly increasing sequence \( k_1, k_2, \ldots \) of nonnegative integers. In this
note we shall prove that if \( \tau \) has a strictly positive subinvariant function
in \( L_\infty(X) \) then the equivalence of (i) and (ii) still holds. Applying this
result, we obtain that if \( T \) is a positive linear operator on \( L_1(X) \) such that
\( \sup_n \|(1/n) \sum_{k=0}^{n-1} T^k\|_1 < \infty \) and also such that \( T^nf \) converges weakly for any
\( f \in L_1(X) \) with \( \int f \, dm = 0 \) and if \( T^* \) has a strictly positive subinvariant
function in \( L_\infty(X) \), then for any \( f \in L_1(X) \) with \( \int f \, dm = 0 \) and any strictly
increasing sequence \( k_1, k_2, \ldots \) of nonnegative integers, \( (1/n) \sum_{i=1}^{n} T^k f \)
converges strongly. This is a generalization of another result of Akcoglu
and Sucheston [1].
2. Results. Throughout this section we shall assume that there exists a strictly positive function $s \in L_\infty(X)$ with $\tau * s \leq s$. In the proofs we shall also assume that $(X, \mathcal{M}, m)$ is a finite measure space, since the $L_1$ of a $\sigma$-finite measure space is isometric to the $L_1$ of a finite measure space (cf. [1]).

**Theorem 1.** The following two conditions are equivalent:

(i) If $f \in L_1(X)$ then $T^n f$ converges weakly;

(ii) If $f \in L_1(X)$ then $(1/n) \sum_{k=1}^{n} T^{k} f$ converges strongly for any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers.

**Proof.** We first prove that (i) implies (ii). For $sf \in L_1(X)$, where $f \in L_1(X)$, define $V(sf) = sTf$. Since $\{sf; f \in L_1(X)\}$ is a dense subspace of $L_1(X)$ in the norm topology and $\|V(sf)\|_1 \leq \|sf\|_1$ (cf. [3]), $V$ may be considered to be a linear contraction on $L_1(X)$. Since $V^n(sf) = s T^n f$ for any $n \geq 0$ and $T^n f$ converges weakly, it follows that $V^n(sf)$ converges weakly. Thus, since $V$ is a contraction, it is easily seen that for any $A \in \mathcal{M}$ the limit $\mu(A) = \lim_n \int_A V^n f \, dm$ exists. Since the measure $m$ is finite, the Vitali-Hahn-Saks theorem implies that $\mu$ is a countably additive measure on $\mathcal{M}$ absolutely continuous with respect to $m$. Therefore there exists a function $g \in L_1(X)$ such that $\mu(A) = \int_A g \, dm$ for any $A \in \mathcal{M}$. It follows that $V^n f$ converges weakly to $g$. Thus, by Theorem 2.1 of [1], for any $f \in L_1(X)$ and any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers,

$$ \frac{1}{n} \sum_{i=1}^{n} V^{k_i} (sf) = \frac{1}{n} s \left( \sum_{i=1}^{n} T^{k_i} f \right) $$

converges strongly. Let $\lim_n \|(1/n)s(\sum_{i=1}^{n} T^{k_i} f) - f_0\|_1 = 0$ for some $f_0 \in L_1(X)$ and let $\varepsilon > 0$ be arbitrarily fixed. Since $T^n f$ converges weakly, there exists a positive number $\delta$ such that $A \in \mathcal{M}$ and $m(A) < \delta$ imply $\int_A |T^n f| \, dm < \varepsilon$ for any $n \geq 0$. Choose $\eta > 0$ such that $m(\{x; s(x) < \eta\}) < \delta$ and $\int_{\{x; s(x) < \eta\}} |f_0| \, dm < \varepsilon$, and put $A = \{x; s(x) < \eta\}$. Then

$$ \left\| \frac{1}{n} \sum_{i=1}^{n} T^{k_i} f - \frac{1}{m} \sum_{j=1}^{m} T^{k_i} f \right\|_1 \leq \left\| \frac{1}{n} \sum_{i=1}^{n} 1_A T^{k_i} f \right\|_1 + \left\| \frac{1}{m} \sum_{j=1}^{m} 1_A T^{k_i} f \right\|_1 $$

$$ + \left\| \frac{1}{n} \sum_{i=1}^{n} 1_{X-A} T^{k_i} f - \frac{1}{m} \sum_{j=1}^{m} 1_{X-A} T^{k_i} f \right\|_1 $$

$$ < 2\varepsilon + \left\| \frac{1}{n} \sum_{i=1}^{n} 1_{X-A} T^{k_i} f - \frac{1}{n} \sum_{i=1}^{n} f_0 \right\|_1 $$

$$ + \left\| \frac{1}{m} \sum_{j=1}^{m} 1_{X-A} T^{k_i} f - \frac{1}{m} \sum_{j=1}^{m} f_0 \right\|_1 . $$
and

\[ \left\| \frac{1}{n} \sum_{i=1}^{n} 1_{X-A} T^{k_i} f - 1_{X-A} \frac{1}{s} f \right\|_1 \leq \frac{1}{\eta} \left\| \frac{1}{n} s \left( \sum_{i=1}^{n} 1_{X-A} T^{k_i} f \right) - 1_{X-A} f \right\|_1 \to 0 \]

as \( n \to \infty \), from which we observe that \((1/n) \sum_{i=1}^{n} T^{k_i} f\) is a Cauchy sequence in \( L_1(X) \), and hence \((1/n) \sum_{i=1}^{n} T^{k_i} f\) converges strongly.

Conversely if (ii) holds, then it follows easily that \( \sup_n \| T^n \|_1 < \infty \) and that for any \( f \in L_1(X) \) and any \( A \in \mathcal{A} \), \( \lim_n \int_A T^n f \, dm \) exists, and hence \( T^n f \) converges weakly. This completes the proof of Theorem 1.

**Theorem 2.** Let \( T \) be a positive linear operator on \( L_1(X) \) with

\[ \sup_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k \right\|_1 < \infty \]

and suppose \( T^s \leq s \) for some \( 0 < s \in L_\infty(X) \). Then the following two conditions are equivalent:

(i) If \( f \in L_1(X) \) and \( \int f \, dm = 0 \), then \( T^n f \) converges weakly;

(ii) If \( f \in L_1(X) \) and \( \int f \, dm = 0 \), then for any strictly increasing sequence \( k_1, k_2, \ldots \) of nonnegative integers, \((1/n) \sum_{i=1}^{n} T^{k_i} f\) converges strongly.

**Proof.** Suppose (i) holds. It is known [3] that if \( T \) has no nontrivial nonnegative invariant function in \( L_1(X) \), then the operator \( V \) introduced above also has no nontrivial nonnegative function in \( L_1(X) \). Thus it follows from [1] that, if \( T^n f \) converges weakly then

\[ \lim_n \| V^n(sf) \|_1 = \lim_n \| sT^n f \|_1 = 0. \]

Let \( \varepsilon > 0 \) be arbitrarily fixed, and let \( \delta \) be a positive number such that \( A \in \mathcal{M} \) and \( m(A) < \delta \) imply \( \int_A |T^n f| \, dm < \varepsilon \) for any \( n \geq 0 \). Choose \( \eta > 0 \) such that \( m(\{ x ; s(x) < \eta \}) < \delta \), and put \( A = \{ x ; s(x) < \eta \} \). Then

\[ \| T^n f \|_1 \leq \| 1_A T^n f \|_1 + \eta^{-1} \| 1_{X-A} sT^n f \|_1 \]

\[ < \varepsilon + \eta^{-1} \| sT^n f \|_1 \]

and \( \| sT^n f \| \to 0 \) as \( n \to \infty \), thus \( \lim_n \| T^n f \|_1 = 0 \).

If there exists \( 0 \leq h \in L_1(X) \) with \( \| h \|_1 > 0 \) and \( Th = h \), then it follows from [1] that for any \( f \in L_1 \), \( T^n f \) converges weakly. Thus the strong convergence of \( (1/n) \sum_{i=1}^{n} T^{k_i} f \) for any strictly increasing sequence \( k_1, k_2, \ldots \) of nonnegative integers follows from Theorem 1.

Clearly (ii) implies (i), and the proof is complete.
Bibliography


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