A GENERALIZATION OF TIETZE'S THEOREM ON CONVEX SETS IN $\mathbb{R}^3$

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Abstract. Let $S \subseteq \mathbb{R}^3$ and let $C(S)$ denote the points of local convexity of $S$. One interesting result which is proven is

Theorem. Let $S \subseteq \mathbb{R}^3$ be such that $S = \text{cl}(C(S))$, $S$ not planar and $C(S)$ is connected. Then $S \subseteq \text{cl}(\text{int } S)$.

1. Introduction. F. A. Valentine in [8] proves that if $S$ is a closed connected subset of $\mathbb{R}^d$ whose points of local nonconvexity are decomposable into $n$ convex sets, then $S$ is $2n+1$ polygonally connected. Guay and Kay in [2] show that if $S$ is a closed connected subset of a topological vector space such that $S$ has exactly $n$ points of local nonconvexity and such that the points of local convexity of $S$ are connected, then $S$ is expressible as a union of $n+1$ or fewer closed convex sets. The purpose of this paper is to give a result which is in the vein of both the latter mentioned results and which generalizes Tietze's theorem on convex sets in $\mathbb{R}^3$. For related results see [1], [2], [3], [4], [5], [6] and [8].

2. Notations and main results. If $S \subseteq \mathbb{R}^d$, the symbols $C(S)$ and $L(S)$ denote the points of local convexity of $S$ and points of local nonconvexity of $S$, respectively. The symbols $\text{int } S$ and $\text{cl } S$ denote the interior of $S$ and the closure of $S$, respectively.

Theorem 1. Let $S \subseteq \mathbb{R}^3$ be such that

1. $S = \text{cl}(C(S))$,
2. $S$ not planar,
3. $C(S)$ is connected.

Then $S \subseteq \text{cl}(\text{int } S)$.

Proof. We first prove $C(S) \subseteq \text{cl}(\text{int } S)$. Suppose not. Then there exists $x \in C(S)$ and an open set $M_x$ about $x$ such that $M_x \cap S$ is convex and $\dim(M_x \cap S) = k < 3$. Let $L$ be the subspace generated by $M_x \cap S$. Let $\mathcal{M} = \{M | M$ is open in $L \cap S, M_x \cap S \subseteq M$ and if $y \in M$, there exists an open set $N_y$ about $y$ such that $N_y \cap S$ is convex and $\dim(N_y \cap S) = k \}$. Note $\mathcal{M} \neq \emptyset$ since $M_x \cap S \subseteq \mathcal{M}$. Partially order $\mathcal{M}$ by set inclusion. Using a standard
Zorn's lemma argument, it may be shown $\mathcal{M}$ has a maximal element $A$. Since $S$ is not planar, there exists $z \in S$, with $z \notin L$. Select a point $q$ as follows: If $z \in C(S)$, set $z=q$. If $z \in L(S)$, since $S \subseteq \text{cl}(C(S))$, there exists a point $r \in C(S)$, with $r \notin L$. Then set $q=r$. Since $C(S)$ is connected and locally convex, $C(S)$ is polygonally connected. Let $l$ be a simple polygonal arc from $x$ to $q$ in $C(S)$. Regarding $x$ as the starting point of $l$, let $m$ be the last point of $l$ in $\text{cl} A$. Since $l \subseteq C(S)$, there exists an open set $N_m$ such that $N_m \cap S$ is convex. It is clear that $\dim(N_m \cap S) \geq k$. We consider two cases.

Case 1. $\dim(N_m \cap S)=k$. Then $N_m \cap S \subseteq L$ and since $N_m \cap S$ contains points of $l$ not in $A$, we have $N_m \cap A \subseteq N_m \cap S$. Then $A \cup (N_m \cap S) \in \mathcal{M}$, contradicting the maximality of $A$.

Case 2. $\dim(N_m \cap S)>k$. Now since $N_m \cap A \neq \emptyset$, we may choose $p \in N_m \cap A$. Then for any open set $N_p$ such that $N_p \cap S$ is convex, $\dim(N_p \cap S) \geq \dim(N_m \cap S)>k$, contradicting that $A \in \mathcal{M}$.

Thus $C(S) \subseteq \text{cl}(\text{int} S)$ and the latter with hypothesis (1) imply the Theorem.

The following theorem is the main result of this paper.

**Theorem 2.** Let $S \subseteq R^3$ be closed, $S$ not planar. Suppose $L(S)$ decomposable into $n$ closed line segments $[a_i b_i]$, $1 \leq i \leq n$. Suppose $C(S)$ is connected and that given $x, y \in C(S)$ that $x$ and $y$ may be joined by an arc $l \subseteq S$ such that $l$ is contained in a hyperplane. Then $S$ is $n+1$ polygonally connected.

**Proof.** The fact that $L(S)$ is decomposable into $n$ closed line segments easily implies that $S \subseteq \text{cl}(C(S))$. Let $x, y \in S$ and let $\mathcal{H}_{xy}$ denote the set of all hyperplanes containing $x$ and $y$. Define a set $F$ by $F=\{(x, y)|(x, y) \in C(S) \times C(S)$ and if $H_{xy} \in \mathcal{H}_{xy}$, $\dim(H_{xy} \cap [a_i b_i]) \leq 0 \forall i, 0 \leq i \leq n\}$, where in the definition of $F$ we take $\dim \emptyset = -1$. Let $(x, y) \in F$. Then by hypothesis there exists $H_{xy} \in \mathcal{H}_{xy}$ and an arc $l \subseteq S$ from $x$ to $y$ such that $l \subseteq H_{xy}$. Let $C$ be the component of $H_{xy} \cap S$ which contains $x$ and $y$. Since $\dim(H_{xy} \cap [a_i b_i]) \leq 0, \forall i$, $C$ has at most $n$ points of local nonconvexity and by a result of Valentine [8], $C$ is $n+1$ polygonally connected. Thus $x$ and $y$ may be joined by an $n+1$ polygonal arc lying in $S$. By Theorem 1, $F$ is dense in $S \times S$, and the theorem follows from a standard limiting argument in the Hausdorff metric.

**Bibliography**


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