METRIC INEQUALITIES AND THE ZONOID PROBLEM

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Abstract. For normed spaces the hypermetric and quasihypermetric properties are equivalent and imply the quadrilateral property. The unit ball of a Minkowski space is a zonoid if and only if the dual space is hypermetric. The unit ball of $L^p_n$ is not a zonoid for $n=3$, $p<\log 3/\log 2$, and for $p\leq 2-(2n \log 2)^{-1}+o(n^{-1})$. The elliptic spaces $\mathbb{R}^d$, $d>1$, are not quasihypermetric.

A metric space $(S, d)$ is said to be hypermetric (Kelly [3]) when

$$\sum_{i,j=1}^{n} w_i w_j d(x_i, x_j) \leq 0$$

for all $n>0$, $x_1, \cdots, x_n$ in $S$, and $w_1, \cdots, w_n$ integers with sum 1. This implies [5] that (1) also holds for real $w_i$ of sum 0, which is called the quasihypermetric property.

A piecewise linear inequality (PLI) is a relation of the form

$$\sum_{i=1}^{k} c_i \left| \sum_{j=1}^{n} a_{ij} x_j \right| \geq 0$$

which holds for all $n$-tuples $x_1, \cdots, x_n$ of real numbers, with fixed real $c_i$ and $a_{ij}$. An example is the quadrilateral inequality [8]

$$|x| + |y| + |z| - |x+y| - |y+z| - |z+x| + |x+y+z| \geq 0.$$

Since the real line is hypermetric [4], (1) generates an infinite family of PLI's of the form

$$\sum_{i,j=1}^{n} (-w_i w_j) |x_i - x_j| \geq 0$$

for $w_i$ integers of sum 1 and for $w_i$ reals of sum 0.

The PLI (2) is said to extend to the normed space $N$ if it holds with the absolute value function replaced by the norm and $x_1, \cdots, x_n$ arbitrary elements of $N$.

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A zonoid [1], [2] is a convex body belonging to the closure (in the Hausdorff set metric) of the class of zonotopes (polytopes which are Minkowski sums of segments).

The theorems of I. J. Schoenberg [7] and P. Lévy [6] imply that the above concepts are related.

**Proposition 1.** For a real normed space $N$ the following 3 properties are equivalent.

(i) every PLI extends to $N$,
(ii) $N$ is quasihypermetric,
(iii) $e^{-\|x\|}$ is positive definite on $N$.

**Proof.** If every PLI extends to $N$ then in particular $N$ is quasihypermetric. Following an argument of Schoenberg, consider $n+1$ points $x_0, \ldots, x_n$ in $N$ with weights $-\sum_{i=1}^n w_i, w_1, \ldots, w_n$ where the $w_i$ are arbitrary reals. This yields

$$
\sum_{i,j=1}^n w_i w_j (\|x_i - x_0\| + \|x_j - x_0\| - \|x_i - x_j\|) \geq 0,
$$

that is, the parenthesis is positive definite. Then its exponential is positive definite and, absorbing $e^{\|x_i - x_0\|}$ into $w_i$, $e^{-\|x\|}$ is shown to be positive definite. Conversely, if $e^{-\|x\|}$ is positive definite on $N$ it is positive definite on every finite dimensional subspace of $N$. By Lévy's theorem [6], [1] these subspaces are isometrically isomorphic to subspaces of $L_1(0, 1)$ to which any PLI extends by integration. Since each PLI involves only finite systems of vectors, it extends to all of $N$.

In [4] Kelly raised the question of the possible relations between the hypermetric and quadrilateral properties in normed spaces. Applying Proposition 1 one has

**Corollary 1.1.** For real normed spaces, the hypermetric and quasihypermetric properties are equivalent and they imply the quadrilateral property.

For $1 \leq p \leq 2$, $e^{-\|x\|}$ is known [7] to be positive definite on $L_p(0, 1)$, hence

**Corollary 1.2.** $L_p(0, 1)$ (and a fortiori $l_p$) is hypermetric and quadrilateral for $1 \leq p \leq 2$.

This had been conjectured by Kelly [4] and the Smileys [8]. For finite dimensional real normed spaces (Minkowski spaces) the positive definiteness of $e^{-\|x\|}$ is equivalent [1] to the property that the unit ball of the dual space is a zonoid. Thus one has

**Corollary 1.3.** The unit ball of a Minkowski space is a zonoid if and only if the dual space is hypermetric.
Thus the known fact that all Minkowski planes are hypermetric [4] follows from the elementary fact that all centrally symmetric convex polygons are sums of segments.

For \( n \geq 3 \), let \( p_n \) be the smallest \( p \) such that the unit ball of \( l^n_p \) is a zonoid. One has \( p_3 \leq p_n \leq p_{n+1} \leq 2 \), Bolker [1], [2] has conjectured that \( p_3 = 2 \). He reports the following bounds of Rosenthal: \( p_3 > \log 9 / \log 7 \) and \( p_n > 2 \log n / \log 3n \), hence \( p_n \geq 2 - \log 9 / \log n + o((\log n)^{-1}) \). These bounds can be substantially improved.

**Proposition 2.** One has \( p_3 \geq \log 3 / \log 2 \) and \( p_n \geq 2 - 1/2n \log 2 + o(n^{-1}) \).

**Proof.** For \( n=3 \), \( p < \log 3 / \log 2 \) the quadrilateral inequality in the dual space is violated for \( x=(1, 1, -1) \), \( y=(1, -1, 1) \), \( z=(-1, 1, 1) \), as observed by the Smiley's [8] 2. For large even \( n=2m \), consider the quasi-hypermetric inequality in the dual \( l^n_2 \), with \( w_i = 1 \) at the \( 2^m \) points with the first \( m \) coordinates equal to \( \pm 1 \) and the last \( m \) coordinates 0, \( w_i = -1 \) at the \( 2^m \) points with first \( m \) coordinates 0 and the last \( m \) equal to \( \pm 1 \). All distances between the two sets are \( (2m)^{1/q} \) while distances within each set are of the form \( 2k^{1/q} \) with \( 0 \leq k \leq m \). Counting the number of occurrences of each distance, a violation of the inequality is seen to require

\[
2 \left( \sum_{k=0}^{m} \binom{m}{k} (2k^{1/q}) \right) > 2^{2m}(2m)^{1/q}
\]

or \( 2E\{k^{1/q}\} > (2m)^{1/q} \) with \( k \) binomially distributed. For large \( m \), expand \( k^{1/q} \) about the mean \( k = m/2 \) and let \( 1/q = 1/2 + \varepsilon \). Then the violation occurs for \( \varepsilon < -(16m \log 2)^{-1} + o(m^{-1}) \), so that \( p_n \geq 2 - (2n \log 2)^{-1} + o(n^{-1}) \) as claimed.

Kelly [3] has shown that spherical spaces are hypermetric. This no longer holds when antipodes are identified.

**Proposition 3.** The elliptic plane \( \mathcal{E}^2 \) is not quasihypermetric.

**Proof.** Assume the opposite, and consider the function, defined for \( \mu \) in \( C(\mathcal{E}^2) \), by \( F(\mu) = \int \mu(dx) \int \mu(dy) \bar{x} \bar{y} \), where \( \bar{x} \bar{y} \) is the elliptic distance and the integrals range over the compact space \( \mathcal{E}^2 \). By (1) the function is nonpositive, hence concave on the subspace \( \{ \mu | \int \mu(dx) = 0 \} \). The concavity holds as well on the parallel subspace \( \{ \mu | \int \mu(dx) = 1 \} \) and in particular on the set \( \mathcal{P} \) of probability measures on \( \mathcal{E}^2 \). For \( \mu \) in \( \mathcal{P} \) and \( \tau \) in the compact group \( G \) of isometries of \( \mathcal{E}^2 \), let \( \mu^* \) be the mixture of the displaced measures \( \mu \circ \tau \) under normalized Haar measure on \( G \). Then \( \mu^* \) is the uniform

1 Thus \( L_p(0, 1) \) is not hypermetric for \( p > 2 \).
2 Alternatively, the hypermetric inequality is violated for the choice of \( w_i = 1 \) at \((\pm 1, \pm 1, 0)\) and \( w_i = -1 \) at \((0, 0, 0), (0, 0, \pm 1)\).
distribution on $\mathcal{S}^2$ and by concavity $F(\mu^*) \geq F(\mu)$. However, the distribution $\mu$ assigning equal probabilities to the vertices of an equilateral triangle of side length $D$, the diameter of $\mathcal{S}^2$, yields $F(\mu) = 2D/3$ while $F(\mu^*) = 2D/\pi$, a contradiction.

That $\mathcal{S}^2$ is not hypermetric already follows from the violation of the hypermetric inequality that occurs for the choice of $w_i = -1$ at 3 mutually orthogonal lines and $w_i = +1$ at their 4 trisectors.

Since $\mathcal{S}^2 \subset E^d$ for $d > 2$ one has

**Corollary 3.1.** For $d > 1$ the elliptic space $\mathcal{S}^d$ is not quasihypermetric.

**References**