FINITE DIMENSIONAL GROUP RINGS

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Abstract. A ring is right finite dimensional if it contains no infinite direct sum of right ideals. We prove that if a group $G$ is finite, free abelian, or finitely generated abelian, then a ring $R$ is right finite dimensional if and only if the group ring $RG$ is right finite dimensional. A ring $R$ is a self-injective cogenerator ring if $R^n$ is injective and $RR$ is a cogenerator in the category of unital right $R$-modules; this means that each right unital $R$-module can be embedded in a direct product of copies of $R$. Let $G$ be a finite group where the order of $G$ is a unit in $R$. Then the group ring $RG$ is a self-injective cogenerator ring if and only if $R$ is a self-injective cogenerator ring. Additional applications are given.

1. Introduction. Let $R$ always denote an associative ring with 1 and $G$ a group with order $|G|$. The group ring of a group $G$ and a ring $R$ is the ring of all formal sums $\sum_{g \in G} r(g)g$ with $r(g) \in R$ and with only finitely many nonzero $r(g)$ [7]. For a right finite dimensional ring $R$, there exists an integer $n$ such that $R$ contains a direct sum of $n$-summands and the number of summands of any other direct sum in $R$ is at most $n$. In this case, we write $\dim R = n$. The ring $R$ will be considered as a right $R$-module $R_R$ and by finite dimensional we shall mean right finite dimensional.

It is known that if $H$ is any semigroup with 1, then $RH$ is a ring. In particular, the polynomial ring is a special case of this construction. Shock has shown that the right finite dimensional property carries over to polynomial rings [10]. This paper extends this result to group rings.

If $R$ is a subring of $Q$ and the identity of $R$ is also the identity of $Q$, then $R$ is a right order in $Q$ if

(a) every nonzero divisor of $R$ is a unit in $Q$, and

(b) every element of $Q$ can be written in the form of $cd^{-1}$ where $c$ and $d$ are in $R$ and $d$ is a nonzero divisor of $R$. We prove that if $G$ is a finite group, then $R$ is a right order in a self-injective cogenerator ring and the order of no finite normal subgroup of $G$ is a zero-divisor in $R$. 

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if and only if $RG$ is a right order in a self-injective cogenerator ring. Let $G$ be a free abelian group. If $R$ is a right order in a right Artinian ring then $RG$ is a right order in a right Artinian ring.

2. Finite dimensional group rings. It is always true that if $RG$ is finite dimensional then $R$ is finite dimensional; however, the converse is not in general true.

Example 2.1. There exists a finite dimensional ring $R$ and a group $G$ such that the group ring $RG$ is not finite dimensional. Let $R$ be a field of characteristic zero and $G=\mathbb{Z}/p\mathbb{Z}$ (for all prime $p$), where $C_p$ is a cyclic group of order $p$. Then $RG$ is not finite dimensional. This follows from the fact that $RG$ is regular and the right ideal $\omega(C_p)$ of $RG$ generated by $\{1-h|h \in C_p\}$ is principal [2]. So the question naturally arises as to when the group ring $RG$ is finite dimensional.

Proposition 2.2 (Shock [10]). A ring $R$ is finite dimensional if and only if the polynomial ring $R[x_1, x_2, \cdots]$ is finite dimensional. Furthermore, $\dim R=\dim R[x_1, x_2, \cdots]$.

Proof. See Theorem 2.6 of [10].

Let $R$ be a subring of $S$, then we call $S$ a ring of right quotients of $R$, if for every $0 \neq s \in S$ and for every $s' \in S$, there exists $r \in R$ such that $sr \neq 0$ and $s'r \in R$. Let $Q(R)$ denote the complete ring of quotients of $R$. It is well known that $R$ is finite dimensional if and only if $Q(R)$ is, and in this case $\dim R=\dim Q(R)$. It is also known that if $S$ is a ring of right quotients of $R$ then $Q(R)$ is the complete ring of quotients of $S$ [4].

Theorem 2.3. Let $G$ be an infinite cyclic group, then $R$ is finite dimensional if and only if $RG$ is finite dimensional. Furthermore, $\dim R=\dim RG$.

Proof. Let $S$ be a multiplicative semigroup isomorphic to the non-negative integers. Then $S$ is a semigroup with identity and is generated by the nonnegative powers of some element, say $g$. By Proposition 2.2, it is clear that $RS$ is finite dimensional, since $RS$ is just a polynomial ring in the variable $g$. Now $S$ can be embedded in an infinite cyclic group $G$, which is generated by all powers of $g$. We need only show that $RG$ is a ring of right quotients of $RS$. Let $r_1, r_2 \in RG$ with

\[
0 \neq r_1 = r_1(g_1)g_1 + \cdots + r_1(g_n)g_n = r_1(g_1)g^{a_1} + \cdots + r_1(g_n)g^{a_n}
\]

and

\[
r_2 = r_2(h_1)h_1 + \cdots + r_2(h_m)h_m = r_2(h_1)g^{b_1} + \cdots + r_2(h_m)g^{b_m}.
\]
Let $k = \max\{|a_i|, |b_j|\}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. It is clear that $r = g^k \in RS$, $r_1 r \neq 0$, and $r a r' \in RS$. Hence, $RG$ is finite dimensional. Also, $\dim Q(RS) = \dim RS = \dim R$ shows that $\dim R = \dim RG$. The converse is clear.

A **free abelian group** is a group which is a direct sum of infinite cyclic groups.

**Corollary 2.4.** Let $G$ be a free abelian group, then $R$ is finite dimensional if and only if $RG$ is finite dimensional. Furthermore, $\dim R = \dim RG$.

**Proof.** Let $H = S_1 \oplus S_2 \oplus \cdots$ where each $S_i$ is a multiplicative semigroup isomorphic to the nonnegative integers. If $R$ is finite dimensional then $RH$ is finite dimensional by Proposition 2.2. Let $G = G_1 \oplus G_2 \oplus \cdots$, where $S_i$ is embedded in the infinite cyclic group $G_i$, and now show that $RG$ is a ring of right quotients of $RH$. The details are omitted. The converse and inequalities follow easily.

**Lemma 2.5.** For a finite group $G$, the group ring $RG$ is finite dimensional if and only if the ring $R$ is finite dimensional. Also, $\dim R \leq \dim RG \leq \dim R \cdot |G|$.

**Proof.** Let $G$ be finite, then $RG_R$ is $R$-isomorphic to a direct sum of $|G|$ copies of the finite dimensional $R$-module $R$. Hence, $RG$ is a finite dimensional $R$-module and therefore a finite dimensional $RG$-module. The converse and inequalities are clear.

**Theorem 2.6.** Let $G$ be a finitely generated abelian group, then $R$ is finite dimensional if and only if $RG$ is finite dimensional. If $H$ is the torsion subgroup of $G$, then $\dim R \leq \dim RG \leq \dim R \cdot |H|$.

**Proof.** If $G$ is a finitely generated abelian group then $G \cong G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus H$ where $|H| < \infty$ and $G_i$ for $1 \leq i \leq n$ is an infinite cyclic group. As in [2, p. 673], we define $A_1 = RG_1$, $A_2 = A_1 G_2$, $\cdots$, $A_n = A_{n-1} G_n$, and $A = A_n H$; clearly $RG \cong A$. By Corollary 2.4 and Lemma 2.5, we see by induction that $A$ is finite dimensional and consequently $RG$ is finite dimensional. The converse and inequalities follow easily.

3. **Applications.** Let $Z(R)$ denote the **right singular ideal** of $R$ (4).

**Lemma 3.1.** Let $G$ be a free abelian group, then $Z(RG) = Z(R)G$.

**Proof.** The proof uses the same technique as the proof of Theorem 2.7 of [10].

**Proposition 3.2 (Connell, [2]).** The group ring $RG$ is semiprime if and only if $R$ is semiprime and the order of no finite normal subgroup is a zero-divisor in $R$.

**Proof.** See the appendix of [4].
It is well known that a semiprime Goldie ring is a semiprime, finite dimensional ring with zero singular ideal.

**Corollary 3.3.** Let $G$ be a free abelian group. A ring $R$ is a semiprime Goldie ring if and only if $RG$ is a semiprime Goldie ring.

**Proof.** The proof is immediate.

**Proposition 3.4 (Burgess, [1]).** If $Z(RG)=0$, then $Z(R)=0$ and the order of every finite normal subgroup of $G$ is a nonzero-divisor in $R$.

**Proof.** See Theorem 4.8 of [1].

A **locally normal group** is one in which every finite subset is contained in a finite normal subgroup.

**Proposition 3.5 (Burgess, [1]).** Assume that $G$ is locally normal and the order of every finite normal subgroup of $G$ is a nonzero-divisor in $R$. If $Z(R)=0$, then $Z(RG)=0$.

**Proof.** See 4.9 of [1].

**Corollary 3.6.** Let $G$ be a finitely generated abelian group. Then $R$ is a semiprime Goldie ring and the order of every finite normal subgroup of $G$ is a nonzero-divisor in $R$ if and only if $RG$ is a semiprime Goldie ring.

**Proof.** The proof is immediate using the construction in the proof of Theorem 2.6.

A right ideal of a ring $R$ is said to be **essential** if it has nonzero intersection with every nonzero right ideal of $R$. A right ideal $D$ of $R$ is **dense** if for every $0 \neq r_1 \in R$ and for every $r_2 \in R$ there exists $r \in R$ such that $r_1 r \neq 0$ and $r_2 r \in D$. We denote the Jacobson radical of $R$ by $\text{Rad } R$. A right ideal $A$ is said to be **small** if for every right ideal $B$, $A + B = R$ implies $B = R$. It is known that $A$ is small if and only if $A \subseteq \text{Rad } R$.

The following remarks are well known.

**Remark 3.7.** A right ideal $D$ is dense in $R$ if and only if $DG$ is dense in $RG$.

**Remark 3.8.** A right ideal $L$ is essential in $R$ if and only if $LG$ is essential in $RG$.

A right ideal $B$ is **rationally closed** in $R$ if $x^{-1}B = \{ r \in R | xr \in B \}$ is not dense for all $x \in R - B$. Let $I(R)$ denote the injective hull of $R$, then $B$ is rationally closed in $R$ if there exists a subset $S$ of $I(R)$ such that $B = \{ x \in R | Sx = 0 \}$ [8].

**Lemma 3.9.** A right ideal $K$ of $R$ is rationally closed in $R$ if and only if $KG$ is rationally closed in $RG$.
Proof. If $K$ is rationally closed then there exists a subset $S \subseteq I(R)$ such that $K = \{ x \in R | Sx = 0 \}$. We will show that $KG = \{ x \in RG | SGx = 0 \}$. Let $x \in KG$ then $SGx = 0$ since $Sk = 0$ for all $k \in K$. Hence $x \in \{ x \in RG | SGx = 0 \}$. Now suppose $0 \not= x \notin KG$. We want to show there exists $y \in SG$ such that $yx \neq 0$. Let $x = r_1(g_1)g_1 + \cdots + r_n(g_n)g_n$, since $x \notin KG$ there exists $r_i(g_i)$ such that $r_i(g_i) \notin K$. $K$ is rationally closed so there exists $0 \neq s \in S$ such that $sr_i(g_i) \neq 0$. Hence, $sx \neq 0$ implies $x \notin \{ x \in RG | SGx = 0 \}$.

Conversely, suppose $K$ is not rationally closed in $R$, then there exists $x \in R - K$ such that $x^{-1}K$ is dense in $R$. Thus $(x^{-1}K)G = x^{-1}KG$ is dense in $RG$ and hence $KG$ is not rationally closed in $RG$.

Proposition 3.10 (Renault, [6]). The group ring $RG$ is self-injective if and only if $R$ is self-injective and $G$ is finite.

Proof. See [6].

Lemma 3.11 (Shock, [9]). Let $R$ be a self-injective ring. Then $R$ is a cogenerator if and only if $R$ is right finite dimensional and $Z(R)$ is rationally closed.

Proof. See Proposition 2 of [9].

If $R$ is a self-injective ring then $Z(R) = \text{Rad } R$ [4]. It is known that if $R$ is self-injective and finite dimensional then $R/\text{Rad } R$ is completely reducible.

Theorem 3.12. Let $G$ be a finite group where the order of $G$ is a unit in $R$, then $R$ is a self-injective cogenerator ring if and only if $RG$ is a self-injective cogenerator ring.

Proof. Let $R$ be a self-injective cogenerator ring. It is clear that $RG$ is finite dimensional and injective. By Lemma 3.11, we need only show that $Z(RG)$ is rationally closed. It is clear that if $R$ contains no proper dense right ideals then every right ideal is rationally closed and conversely. So, we shall show that $RG$ contains no proper dense right ideals. Let $D$ be a dense right ideal of $RG$. Then $D + Z(R)G$ is dense and by Proposition 5.1 of [8], $(D + Z(R)G)/Z(R)G$ is dense in $RG/Z(R)G$ since $Z(R)G$ is rationally closed. Clearly, $RG/Z(R)G$ and $R/Z(R)$ are completely reducible. Therefore, $(R/Z(R))G \simeq RG/Z(R)G$ is completely reducible [2] and thus $RG/Z(R)G$ contains no proper dense right ideals. Hence, $D + Z(R)G = RG$. But $Z(R)G \subseteq Z(RG) = \text{Rad } RG$ implies $Z(R)G$ is small. Hence, $D = RG$. Conversely, let $D$ be dense in $R$, $D \neq R$, then $DG$ is dense in $RG$ and $DG \neq RG$. 

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Lemma 3.13 (Shock, [9]). Suppose that $Z(Q(R))$ is the Jacobson radical of $Q(R)$ and is rationally closed. If $Q(R)/Z(Q(R))$ is a completely reducible ring and $R/Z(R)$ is semiprime, then $R$ is a right order in $Q(R)$.

Proof. See Proposition 4 of [9].

Theorem 3.14. Let $G$ be a finite group, then $R$ is a right order in a self-injective cogenerator ring and the order of no finite normal subgroup of $G$ is a zero-divisor in $R$ if and only if $RG$ is a right order in a self-injective cogenerator ring.

Proof. Let $R$ be a right order in a self-injective cogenerator ring $Q$, then $Q=Q(R)$. By 3.6 of [1], we have $Q(RG)\cong Q(R)G$ and thus by Theorem 3.12 $Q(RG)$ is a self-injective cogenerator ring. It is now clear that both $Q(RG)/Z(Q(RG))$ and $Q(R)/Z(Q(R))$ are completely reducible. Also, it is clear that $Q(R)G/Z(Q(R))G$ is completely reducible and that $RG/Z(R)G$ is semiprime. By Lemma 3.13 we need only to show that $RG/Z(RG)$ is semiprime. To do this, we first show that $Z(R)G=Z(RG)$. It is sufficient to show that $Z(Q(RG))=Z(Q(R))G$ since $Z(RG)=Z(Q(RG))\cap RG=Z(Q(R)G)\cap RG=Z(Q(R))G\cap RG=Z(R)G$. Now $(Q(R)/(Z(Q(R))))G\cong Q(R)G/Z(Q(R))G\cong Q(RG)/Z(Q(RG))$. Recall $Z(Q(R))G\subseteq Z(Q(RG))=\text{Rad }Q(RG)$. Hence, $Z(Q(R))G=Z(Q(RG))$ since $Q(RG)/Z(Q(RG))G$ is completely reducible. The converse follows similarly.

In [12] Smith showed that if $G$ is a poly- (cyclic or finite) group and $R$ is a right order in a right Artinian ring then $RG$ is a right order in a right Artinian ring. We extend this result to a class of group rings, where $G$ need not be poly- (cyclic or finite), using a method of Small [11].

Theorem 3.15. Let $G$ be a free abelian group. If $R$ is a right order in a right Artinian ring then $RG$ is a right order in a right Artinian ring.

Proof. It is clear that $\text{rad }RG=(\text{rad }R)G$ when $G$ is free abelian. We now use the same argument as in Theorem 3.6 of [10].

References


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