TANGENTIAL ASYMMETRIC VALUES OF BOUNDED ANALYTIC FUNCTIONS

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Abstract. Suppose \( f \) is a bounded analytic function on the unit disc whose Fatou boundary function is approximately continuous from above at 1 with value 0. It is well known that \( f \) tends to zero radially and therefore along every nontangential arc. Tanaka [3] and Boehme and Weiss [1] have shown that \( f \) must also tend to zero along certain arcs which are tangential from above. The purpose of this paper is to improve their results by producing a larger collection of such tangential arcs along which \( f \) tends to zero. We construct a class of examples to show that our result is actually better.

1. Introduction and main theorem. Let \( f \) be a bounded analytic function on the unit disc \( D = \{re^{i\theta} : r < 1\} \). By a classical theorem of Fatou, \( f \) has radial limits for almost all \( \theta \). We denote these limits by \( f(e^{i\theta}) \) and call this function the (Fatou) boundary function of \( f \). Given \( \varepsilon > 0 \) let

\[
S_\varepsilon = S_\varepsilon(f) = \{ e^{i\theta} : 0 \leq \theta \leq \pi, |f(e^{i\theta})| < \varepsilon \}
\]

and let \( S_\varepsilon' \) denote the complement of this set in the upper semicircle. Then, by definition, \( f(e^{i\theta}) \) is approximately continuous from above at 1 with value 0 if

\[
\lim_{\varepsilon \to 0+} \int_0^\pi \chi_{S_\varepsilon'}(e^{i\theta}) \, d\theta = 0
\]

for every \( \varepsilon > 0 \), where \( \chi_M \) denotes the characteristic function of the set \( M \).

If we let \( \delta(\theta) = 0^{-1} \int_0^\theta |f(e^{i\tau})| \, d\tau \), then it is easily seen that \( f \) is approximately continuous from above at 1 with value 0 if and only if \( \lim_{\delta \to 0+} \delta(\theta) = 0 \).

Supposing that \( \lim_{\theta \to 0+} \delta(\theta) = 0 \), Tanaka [3] proved that \( f(re^{i\theta}) \) tends to zero along any curve for which \( 2\theta(\delta(2\theta))^{1/2}/(1-r) \) remains bounded.

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This was improved in [1, Theorem 3.1], where it is shown that \( f(re^{i\theta}) \) still tends to zero along any curve for which \( 2\theta\delta(2\theta)/(1-r) \) tends to zero. Using Jensen’s inequality and Lemma 3.2 of [1] we obtain the following theorem.

**Theorem 1.** Let \( f \) be a bounded analytic function on the unit disc with boundary function \( f(e^{i\theta}) \). Suppose \( f(e^{i\theta}) \) is approximately continuous from above at 1 with value 0. Then, \( f \) tends to zero along any upper tangential curve \( \Gamma \) for which

\[
\limsup_{re^{i\theta} \to 1, re^{i\theta} \in \Gamma} \frac{1}{1-r} \int_0^{2\theta} \chi_{S'_e}(e^{it}) \, dt = o(\log 1/\epsilon)
\]

as \( \epsilon \to 0 \).

**Proof.** Lemma 3.2 of [1] proves that for any measurable subset \( S \) of \( \{e^{it}: 0 \leq t \leq \pi\} \) and all small enough \( \theta > 0 \),

\[
u_S(re^{i\theta}) \geq \frac{2}{\pi} \tan^{-1}\left(\frac{a/2\theta}{(1-r)/\tan(\theta/2) + 4\theta[1-(a/2\theta)]/(1-r)}\right)
\]

where \( \nu_S \) is the harmonic measure of \( S \) and

\[
a = a(S, \theta) = \int_0^{2\theta} \chi_{S}(e^{it}) \, dt.
\]

Thus, if \( \Gamma \) is any upper tangential arc in \( D \) terminating at 1 and if \( a(S, \theta)/2\theta \) tends to 1 as \( \theta \to 0^+ \), then

\[
\liminf_{re^{i\theta} \to 1, re^{i\theta} \in \Gamma} \nu_S(re^{i\theta}) \geq \frac{2}{\pi} \tan^{-1}\left(1/2 \limsup_{re^{i\theta} \to 1, re^{i\theta} \in \Gamma} \frac{a(S', \theta)}{1-r}\right).
\]

We assume now without loss of generality that \( |f(e^{i\theta})| < 1 \). Using Jensen’s inequality we then obtain the estimates for \( re^{i\theta} \in \Gamma, \theta \to 0, \) and \( 0 < \epsilon < 1 \),

\[
\limsup \log |f(re^{i\theta})| \leq \limsup \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| P_r(\theta - t) \, dt
\]

\[
\leq \limsup \frac{1}{2\pi} \int_{\chi_{S_e}} \log |f(e^{it})| P_r(\theta - t) \, dt
\]

\[
\leq \log \epsilon \liminf \nu_{S'_e}(re^{i\theta})
\]

\[
\leq \log \epsilon \frac{2}{\pi} \tan^{-1}\left(1/2 \limsup \frac{a(S'_e, \theta)}{1-r}\right)
\]

where \( P_r(\theta) \) is the Poisson kernel. It is easily seen that our assumption implies that the last member of the above string of inequalities tends to \( -\infty \) along \( \Gamma \). Consequently, \( f \) must tend to zero along \( \Gamma \) as claimed.

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If we now return to the condition of Theorem 3.1 of [1] we see that it implies the condition of Theorem 1. To verify this, suppose \(2\theta \delta(2\theta)/(1-r)\) tends to zero along the upper tangential curve \(\Gamma\). Then, for any \(\varepsilon > 0\),

\[
\varepsilon \cdot \frac{1}{1-r} \int_0^{2\theta} \chi_{S_\varepsilon}(e^{it}) \, dt \leq \frac{1}{1-r} \int_0^{2\theta} |f(e^{it})| \, dt = \frac{2\theta \delta(2\theta)}{1-r}.
\]

Therefore, for any \(\varepsilon > 0\), \(\lim \sup a(S_\varepsilon, \theta)/(1-r) = 0\) along \(\Gamma\) and the condition of Theorem 1 above is easily satisfied.

2. A class of examples. If \(f(e^{i\theta})\) is actually continuous from above at 1 with value 0, then for every \(\varepsilon > 0\), \(S_\varepsilon\) includes an interval abutting 1 from above. Thus, the condition of Theorem 1 is satisfied for any upper tangential curve at 1 as one would hope. This is not the case for the result of [1]. It does not prove, for example, that such a continuous function must tend to zero along the upper tangential curve \(2\theta \delta(2\theta) = 1-r\).

In this section we will construct a class of functions which demonstrates further the distance between Theorem 3.1 of [1] and Theorem 1 of this paper. Before proceeding to the construction of these examples we require some preliminaries.

Throughout the remainder of the paper \(\rho(\theta)\) will denote a nondecreasing convex function defined near \(0^+\) with

\[
\rho(0) = \rho'(0) = 0, \quad \rho(\theta) > 0 \quad \text{for} \quad \theta \neq 0.
\]

**Lemma 1.** With \(\rho\) as above and \(c\) any constant

\[
\lim_{\theta \to 0^+} \frac{\rho(\theta + c\rho(\theta))}{\rho(\theta)} = 1.
\]

The proof of this lemma is not hard and can be found in [2].

**Lemma 2.** Let \(\rho\) be as above. Let \(c\) be a positive constant and suppose \(\theta_1 > \theta_2 > \cdots, \theta_n \to 0^+\) are defined so that the system of intervals \([\theta_n - c\rho(\theta_n), \theta_n + c\rho(\theta_n)]\), \(n = 1, 2, \cdots\), is nonoverlapping. Suppose further that, as \(n \to \infty\), \(\sum_{m=n+1}^{\infty} \rho(\theta_m) = o(\rho(\theta_n))\). Then, if \(I\) is the union of the above system of intervals,

\[
\lim_{\theta \to 0^+} \frac{1}{\rho(\theta)} \int_0^\theta \chi_I(e^{it}) \, dt = 2c.
\]

**Proof.** Let \(F(\theta) = (1/\rho(\theta)) \int_0^\theta \chi_I(e^{it}) \, dt\). Clearly, when \(\theta_{n+1} + c\rho(\theta_{n+1}) \leq \theta \leq \theta_n - c\rho(\theta_n)\), we have \(F(\theta) \leq F(\theta_{n+1} + c\rho(\theta_{n+1}))\). Thus, \(\lim \sup F(\theta)\)
can be computed restricting θ to lie in the intervals of I. If 0n - cr(θn) ≤ θ ≤ θn + cr(θn), then

\[ F(θ) = \left[ θ - (θ_n - cr(θ_n)) + \sum_{m=n+1}^{∞} 2cr(θ_m) \right] / cr(θ) \]

\[ \leq \left[ 2cr(θ_n) + \sum_{m=n+1}^{∞} 2cr(θ_m) \right] / cr(θ - cr(θ_n)) \]

\[ = \frac{ρ(θ_n + cr(θ_n))}{ρ(θ_n) - cr(θ_n))} \frac{ρ(θ_n)}{ρ(θ)} F(θ_n + cr(θ_n)). \]

From Lemma 1 and our hypotheses we have

\[ \limsup_{θ→0^+} F(θ) = \limsup_{n→∞} F(θ_n + cr(θ_n)) \]

\[ = \limsup_{n→∞} \frac{2cr(θ_n) + \sum_{m=n+1}^{∞} 2cr(θ_m)}{ρ(θ_n + cr(θ_n))} \]

\[ = 2c. \]

**Example.** Let ρ be a convex function as above, let Fρ be the convex upper tangential curve \( 1 - r = ρ(θ) \). Then, there exists a corresponding bounded analytic function f such that, as ε→0⁺,

\[ \limsup_{ε→0⁺} \frac{1}{ε} \int_0^{2θ} \chi_{ε}(e^{it}) dt = o(\log 1/ε) \]

while

\[ \limsup_{ε→0⁺} \frac{1}{ε} \int_0^{2θ} \chi_{ε}(e^{it}) dt = \infty \]

so that \( \limsup_{θ→0^+} 2θδ(2θ)/(1-r) \neq 0 \) as \( θ→0^+ \).

**Proof.** The example is constructed in several stages. For fixed ε ∈ (0, 1/ε) let \( m(ε) \) denote the greatest integer in \( (\log 1/ε)^{1/2} \). Choose a nondecreasing sequence \( \{a(k)\} \) of positive numbers tending to ∞ but so slowly that

\[ \sum_{k=1}^{m(ε)} a(k) \leq (\log 1/ε)^{3/4} \]

for all \( 0 < ε < 1/ε \).

For example, choose \( \{a(k)\} \) so that the average of the first \( N \) elements is less than \( √{N}/2 \).

We construct below a doubly infinite sequence of nonoverlapping intervals \( I_{n,k} = [θ_{n,k} - a(k)ρ(θ_{n,k}), θ_{n,k} + a(k)ρ(θ_{n,k})] \), \( n, k = 1, 2, 3, \cdots \) in such a way that, for each \( k \),

\[ \sum_{m=n+1}^{∞} ρ(θ_{m,k}) = o(ρ(θ_{n,k})), \quad \text{as } n → ∞. \]
We, furthermore, require that if $J_0, J_1, \cdots$ are the intervals complementary to the system $\{I_{n,k}\}$ in $(0, \pi/2)$, then the right-hand endpoints $\varphi_0, \varphi_1, \cdots$ of $J_0, J_1, \cdots$ proceed to 0 at least as fast as a geometric sequence so that

$$
\sum_{k=1}^{\infty} k^n \varphi_k < \infty.
$$

Explicitly, choose $\theta = \frac{1}{2}$ and positive values of $\theta_n$ such that $\theta_n \varphi < \theta_n/4^n$ and $\rho(\theta_n) < \rho(\theta_n)/4^n$. Let $\theta_n = \theta_n z (2n-1)$, $n, k = 1, 2, 3, \cdots$; and let

$$
\Phi_n = \{ \varphi : \varphi - a(k) \rho(\varphi) \leq 0 \leq \varphi + a(k) \rho(\varphi) \}.
$$

For fixed $k$ and $m$ and $n = 1, 2, 3, \cdots$,

$$
\frac{\theta_n - a(k) \rho(\theta_n)}{\theta_n + a(m) \rho(\theta_n)} > \frac{\theta_n - a(k) \rho(\theta_n)}{\theta_n/4^n + a(m) \rho(\theta_n/4^n)}
$$

$$
= \frac{4^n - a(k) \rho(\theta_n)/\theta_n}{1 + a(m) \rho(\theta_n/4^n)/(\theta_n/4^n)} \to \infty \text{ as } n \to \infty,
$$

since $\rho(\theta) = 0$. Because of this we may choose $n_1$ so large that the intervals $\Gamma_{n_1,1}, \Gamma_{n_1+1,1}, \cdots$ are nonoverlapping. Call these intervals $I_{1,1}, I_{2,1}, \cdots$ and denote their centers by $\theta_{1,1}, \theta_{2,1}, \cdots$. This done, we may choose $n_2$ so large that the intervals $I_{1,1}, I_{2,1}, \cdots$ and $I_{n_2,2}, I_{n_2+1,2}, \cdots$ are nonoverlapping and so that $I_{n_2,2}$ is closer to zero than $I_{1,1}$. Rename the latter (primed) intervals $I_{1,2}, I_{2,2}, \cdots$ and their centers $\theta_{1,2}, \theta_{2,2}, \cdots$. Continuing inductively we obtain a system $I_{n,k}$ of nonoverlapping subintervals of $(0, \pi/2)$. It is also clear that the endpoints $\varphi_0, \varphi_1, \cdots$ of the complementary intervals to this system proceed to zero at least as rapidly as the midpoints $\theta_n$ of the original system. Since these latter are geometric, condition (5) above is satisfied. To verify condition (4), we first estimate

$$
\sum_{m=n+1}^{\infty} \rho(\theta_m) < \sum_{j=1}^{\infty} \frac{1}{4^j + j(j-1)/2} < \frac{1}{4^n} \to 0 \text{ as } n \to \infty.
$$

Then, using the identity $\theta_{m,k} = \theta_{(m+n_k-1),k} = \theta_{(2n+2n_k-3)}$, which follows from our construction, we have for fixed $k$

$$
\sum_{m=n+1}^{\infty} \rho(\theta_{m,k}) < \sum_{m=2k(2n+2n_k-1)}^{\infty} \rho(\theta_{m,k}) \to 0 \text{ as } n \to \infty.
$$

where $N = 2k(2n+2n_k-1)$.

The next step is to define the function $f$ of the example. For $k = 1, 2, \cdots$ let $I_k = \cup_{n=1}^{\infty} I_{n,k}$. Define $k(\theta)$ for $0 \leq \theta \leq 2\pi$ by setting $k(\theta) = 0$ for $\theta \in [\pi/2, 2\pi]$ and for $\theta \in J_0$. For $k = 1, 2, \cdots$ let $k(\theta) = -k^2$ for $\theta \in I_k \cup J_k$. 

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Since by condition (5) above $-\int_0^{\theta} k(\theta) \, d\theta \leq \sum_{k=1}^{\infty} k^2 \varphi_k < \infty$, we have that $k(\theta)$ is integrable. Therefore, the function

$$f(z) = \exp \frac{1}{2\pi} \int_0^{2\pi} e^{it} + \frac{z}{e^{it} - z} \, k(t) \, dt$$

is bounded analytic on $D$ and as is well known, $|f(e^{i\theta})| = e^{k(\theta)}$ a.e.

We now see that for $0 < \varepsilon < 1/e$, $|f(e^{i\theta})| \geq \varepsilon$ if and only if $e^{k(\theta)} \geq \ln \varepsilon$. Since the values of $k(\theta)$ are $0, -1, -4, -9, \cdots$ we have $-k^2 \geq \ln \varepsilon$ if and only if $k \leq m(\varepsilon)$. Thus, for each $0 < \varepsilon < 1/e$ there is an interval $K_\varepsilon$ abutting 0 on the right such that $S'_{\varepsilon} \cap K_\varepsilon$ consists exactly of all but a finite number of the intervals of $\bigcup_{k=1}^{m(\varepsilon)} I_k$.

We are now in a position to verify the properties (1) and (2) of the example. Recalling Lemma 2 and the definitions of $a(k), m(\varepsilon), S'_{\varepsilon}, T_{\rho}$, and $I_k$ we make the following estimates for $\theta \to 0^+$, $0 \in K_\varepsilon$ and $re^{i\theta} \in T_{\rho}$,

$$2a(m(\varepsilon)) = \limsup \frac{1}{\rho(2\theta)} \int_0^{2\theta} X_{I_{m(\varepsilon)}}(e^{it}) \, dt$$

$$= \limsup \frac{1}{1 - r} \int_0^{2\theta} X_{I_{m(\varepsilon)}}(e^{it}) \, dt$$

$$\leq \limsup \frac{1}{1 - r} \int_0^{2\theta} X_{S'_{\varepsilon}}(e^{it}) \, dt$$

$$= \limsup \sum_{k=1}^{m(\varepsilon)} \frac{1}{\rho(2\theta)} \int_0^{2\theta} X_{I_k}(e^{it}) \, dt \leq 2 \sum_{k=1}^{m(\varepsilon)} a(k)$$

$$\leq (\log 1/\varepsilon)^{3/4} = o(\log 1/\varepsilon).$$

Finally, the assertion following (2) of the example is seen from the computations of the last paragraph of §1.

REFERENCES


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