COMPLEX ITERATED RADICALS

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ABSTRACT. We prove the convergence of the sequence $S$ defined by $z_{n+1} = (z_n - c)^{1/2}$, $c$ real, for any choice of $z_0$. Let $k = |\frac{1}{2} - c|^{1/2}$. If $c < 0$ or $c = \frac{1}{2}$, $S$ has only one fixed point $w = \frac{1}{2} + k$ and converges to $w$ for any $z_0$. If $0 \leq c < \frac{1}{2}$, $S$ has the fixed points $w_1 = \frac{1}{2} + k$ and $w_2 = \frac{1}{2} - k$, and for any $z_0 \neq w_2$, $S$ converges to $w_1$. If $c > \frac{1}{2}$, $S$ has the fixed points $w_1 = \frac{1}{2} + ik$ and $w_2 = \frac{1}{2} - ik$ and converges to $w_1$ if $\Re(z_0) \geq 0$ and to $w_2$ otherwise. We show that convergence is strictly monotone when the neighborhood system is the pencil of coaxial circles with $w_1$ and $w_2$ as limiting points, and give rates of convergence.

The purpose of this paper is to prove that the sequence of complex numbers defined by $z_{n+1} = (z_n - c)^{1/2}$, $c$ real, converges for any choice of $z_0$, i.e. globally, and to discuss the limit and rate of convergence. This problem was posed by C. S. Ogilvy [1].

As a guide to global convergence we first investigate local convergence. The following facts are known about the sequence defined by $z_{n+1} = f(z_n)$ for some choice of $z_0$.

**Lemma 1.** If $\lim_{n \to \infty} z_n = w$ and $f$ is continuous at $w$ then $f(w) = w$, i.e. $w$ is a fixed point.

**Proof.** $w = \lim_{n \to \infty} z_n = \lim_{n \to \infty} z_{n+1} = \lim_{n \to \infty} f(z_n) = f(w)$.

Suppose $w$ is a fixed point of $f$. If $z_N = w$ for some $N$, then $z_n = w$ for all $n \geq N$ and we call the sequence trivial. We define

$$q(n, w) = |z_{n+1} - w|/|z_n - w|.$$

**Lemma 2.** Suppose $w$ is a fixed point of $f$ and $f'(w)$ exists. If $|f'(w)| < 1$, then for $z_0$ sufficiently close to $w$ the sequence converges to $w$ and for all nontrivial sequences $\lim_{n \to \infty} q(n, w) = |f'(w)|$ and we say that $|f'(w)|$ is the local rate of convergence. If $|f'(w)| > 1$, then the sequence cannot converge to $w$ except trivially.

Received by the editors June 15, 1972 and, in revised form, February 13, 1973.


Key words and phrases. Iterated radicals, convergence, rate of convergence, coaxial circles.

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The result for trivial sequences is immediate. Otherwise \[ q(n, w) = \frac{|z_{n+1} - w|}{|z_n - w|} = \frac{|f(z_n) - f(w)|}{|z_n - w|}. \] If \( |f'(w)| = 1 - 2\varepsilon \) for some \( \varepsilon > 0 \), then \( q(n, w) < 1 - \varepsilon \) for \( z_n \) in some deleted circular neighborhood \( \mathcal{M} \) of \( w \) by the definition of derivative. Since \( \mathcal{M} \) is circular, \( z_{n+1} \) also lies in \( \mathcal{M} \). For \( z_0 \) in \( \mathcal{M} \) it follows that \( |z_n - w| < |z_0 - w|(1 - \varepsilon)^n \). The conclusion is immediate.

If \( |f'(w)| = 1 + 2\varepsilon \) for some \( \varepsilon > 0 \) then \( q(n, w) > 1 + \varepsilon \) for \( z_n \) in some deleted neighborhood \( \mathcal{M} \) of \( w \) so that the sequence is not eventually in \( \mathcal{M} \).

We now focus on the sequence defined by \( z_{n+1} = (z_n - c)^{1/2} \) where the argument of the square root lies in \( (-\frac{1}{2}\pi, \frac{1}{2}\pi] \). We note that \( \text{Re}(z_{n+1}) \geq 0 \) for \( n \geq 0 \) and that if \( w \) is a fixed point, \( z_N = w \) implies \( z_{N-1} = w \) and thus \( z_0 = w \) so we exclude the cases \( \text{Re}(z_0) < 0 \) and \( z_0 = w \) from further discussion.

We define \( k = |\frac{1}{2} - c^{1/2}|. \)

**Lemma 3.** If \( c < \frac{1}{4}, \) the sequence converges locally to \( w_1 = \frac{1}{2} + k \) at the rate \( (1 + 2k)^{-1} \). For \( 0 \leq c < \frac{1}{4}, \) \( w_2 = \frac{1}{2} - k \) is the other fixed point but the sequence cannot converge to \( w_2 \).

If \( c > \frac{1}{4}, \) the sequence converges locally to \( w_1 = \frac{1}{2} + ki \) and \( w_2 = \frac{1}{2} - ki \) at the rate \( (1 + 4k^2)^{-1/2} \).

**Proof.** The roots of \( w^2 = w - c \) are \( w_1 = (\frac{1}{2} - c)^{1/2} \) and \( w_2 = (\frac{1}{2} - c)^{1/2} \) and these are the roots of \( w = (w-c)^{1/2} \) except when \( c < 0 \) in which case \( w_2 < 0 \). Since \( f(z) = (z-c)^{1/2}, f'(w) = \frac{1}{2}(w-c)^{1/2} = \frac{1}{2}w. \)

If \( c < \frac{1}{4}, f'(w_1) = (1 + 2k)^{-1} < 1. \)

If \( 0 < c < \frac{1}{4}, f'(w_2) = (1 - 2k)^{-1} > 1. \)

If \( c > \frac{1}{4}, |f'(w_1)| = |f'(w_2)| = \frac{1}{2}c^{1/2} = (1 + 4k^2)^{-1/2} < 1. \)

We shall now prove global convergence and obtain rates very nearly equal to the local rates of convergence. We first consider two simple cases.

**Theorem 1.** For \( c < 0, \) the sequence converges to \( w_1 \) for all \( z_0 \) and \( q(n, w_1) \leq (\frac{1}{4} + k)^{-1}. \) For \( c > 1, \) the sequence converges to \( w_1 \) if \( \text{Im}(z_0) \geq 0 \) and to \( w_2 \) if \( \text{Im}(z_0) < 0. \) In each case \( q(n, w) \leq (\frac{1}{4} + k^2)^{-1/2}. \)

**Proof.** For any \( c \) and either \( w, \)

\[ 1/q(n, w) = |z_n - w|/|z_{n+1} - w| = |z_{n+1} + w|. \]

For \( c < 0, \) \( w_1 = \frac{1}{2} + k \) is real and \( \text{Re}(z_{n+1}) \geq 0 \) so \( |z_{n+1} + w_1| \geq w_1. \) For \( c > \frac{1}{4} \) and a fortiori for \( c > 1, \) \( w_1 = \frac{1}{2} + ki \) lies in the first quadrant and if \( z_0 \) lies in the closed first quadrant so does \( z_n \) for each \( n \) so

\[ |z_{n+1} + w_1| \geq |w_1| = c^{1/2} = (\frac{1}{4} + k^2)^{1/2}. \]

The argument is similar if \( \text{Im}(z_0) < 0. \)
The factor in the first case is twice the local rate; the factor in the second case will be improved in Theorem 3.

If $\frac{1}{2} < c < 1$, $z_n = c + 2\varepsilon^2 i$, and $\varepsilon > 0$ is sufficiently small, then

$$|z_{n+1} + w| = |\varepsilon + \varepsilon i + w| < |w| + 2\varepsilon = c^{1/2} + 2\varepsilon < 1.$$ 

If $0 < c < \frac{1}{2}$ and $c < z_n < w_2$, then $0 < z_{n+1} < z_n$ and again $q(n, w) > 1$. Since, as we shall prove, every sequence converges, we see that a global convergence factor cannot be expressed in terms of the usual metric in which the distance of $z_n$ from $w$ is the radius of the circle passing through $z_n$ with $w$ as center. Instead, for each value of $c$, we set up a metric based on pencils of coaxial circles with $w_1$ and $w_2$ as limiting points.

For $0 \leq c \leq \frac{1}{4}$ we have $w_1 = \frac{1}{2} + k$ and $w_2 = \frac{1}{2} - k$ where $0 \leq k \leq \frac{1}{2}$. The centers of these circles lie on the real axis outside the interval $(\frac{1}{2} - k, \frac{1}{2} + k)$ and the circle with center $\frac{1}{2} \pm k$ has radius $R$ where $R^2 + k^2 = H^2$. The line $x = \frac{1}{2}$ is the radical axis of the pencil. For $c = \frac{1}{4}$, $w_1 = w_2 = \frac{1}{2}$ and the circles are tangent to $x = \frac{1}{2}$ at $(\frac{1}{2}, 0)$. For each $z$ not on the radical axis there is precisely one circle $C(z)$ of the pencil passing through $z$. These are the only properties of the pencil we require. We define $d(z, w_1)$ to be the radius of $C(z)$ if $\text{Re}(z) > \frac{1}{2}$ and $\infty$ otherwise, and $d(z, w_2)$ to be the radius of $C(z)$ if $\text{Re}(z) < \frac{1}{2}$ and $\infty$ otherwise. Notice that $d(z, w)$ and $|z - w|$ are asymptotic for $z$ near $w$ if $c \neq \frac{1}{4}$.

**Theorem 2.** For $0 \leq c \leq \frac{1}{4}$, $d(z_n, w_2)$ is a strictly increasing function of $n$ as long as it is finite and $d_n = d(z_n, w_1)$ is eventually finite. Then $d_n$ is strictly decreasing to zero. In fact

$$d_{n+1} \leq (\frac{1}{2}d_n)^{1/2} \quad \text{for } d_n \geq \frac{1}{2},$$

$$1/d_{n+1} \geq 1/d_n + 1 \quad \text{for } d_n \leq 1,$$

$$1/d_{n+1} \geq 1/d_n + 2 - 4d_n \quad \text{for } d_n \leq \frac{1}{2},$$

$$d_{n+1} \leq (1 + 4k)^{-1/2}d_n \quad \text{for } c < \frac{1}{4} \text{ and all } d_n < \infty.$$ 

**Comments.** For $c = \frac{1}{4}$ the next to the last inequality gives the correct asymptotic rate of convergence. For $0 \leq c < \frac{1}{4}$ the global convergence factor of $(1 + 4k)^{-1/2}$ is never more than $2 \cdot 3^{-1/2} \approx 1.15$ times the local convergence factor $(1 + 2k)^{-1}$. When we restrict $\text{Re}(z_n) \geq \frac{1}{2}$ the global factor for $c < 0$ of Theorem 1 improves to $(1 + k)^{-1}$.

**Proof.** Suppose $d(z_{n+1}, w_2) = R < \infty$ i.e.

$$z_{n+1} = \frac{1}{2} - H + R \cos \theta + iR \sin \theta = u + iR \sin \theta$$

where $H = (R^2 + k^2)^{1/2}$ and $0 \leq u < \frac{1}{2}$. Let $r = aR$ and $h = (r^2 + k^2)^{1/2}$. We shall show for the proper choice of $a < 1$ that $|z_n - (\frac{1}{2} - h)| < r$. i.e. $d(z_n, w_2) < r$. 

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This implies that \(d(z_{n+1}, w_2) > a^{-1}d(z_n, w_2)\). In fact

\[
|z_n - (\frac{1}{2} - h)|^2 - r^2 = |z_{n+1}^2 + c - \frac{1}{2} + h|^2 - r^2 \\
= |(u + iR \sin \theta)^2 - k^2 - \frac{1}{4} + h|^2 - r^2 \\
= |(2uR \cos \theta + h - H) + 2uR \sin \theta|^2 - r^2,
\]

after substituting for \(u^2\) and \(k^2 = H^2 - r^2\),

\[
(1) \quad R^2(u + \frac{1}{2}) - \frac{R^2(1 - a^2)(u + H)}{H + h} > R^2[u + \frac{1}{2} - (1 - a^2)(u + \frac{1}{2})/2k], \quad \text{for } H \leq \frac{a}{2},
\]

\[
> 0 \quad \text{for } a = (1 - 4k/3)^{1/2}.
\]

Thus for \(0 \leq \epsilon < \frac{1}{4}\) the relation \(d(z_{n+1}, w_2) > (1 - 4k/3)^{-1/2}d(z_n, w_2)\) is valid until \(H\) exceeds \(\frac{a}{2}\). If \(\epsilon = \frac{1}{4}\) then \(H = R\) and \(h = r = a\) and the second factor rearranges to \(R[(a + R - 1)u + (a - \frac{1}{2})R]\) which is positive for \(a = \max(1 - R, \frac{1}{2})\). Here too \(H\) eventually exceeds \(\frac{a}{2}\). Now the preimage of the line \(x = \frac{1}{2}\) under the map \(f(z) = (z - c)^{1/2}\) is the parabola \(x = c + \frac{1}{2} - y^2\) and the portion of this parabola in the right half-plane is interior to the circle corresponding to \(H = \frac{a}{2}\), i.e. the circle with center \(-\frac{1}{2}\) and radius \((c + 5/16)^{1/2}\). Thus once an iterate has a value of \(H\) exceeding \(\frac{a}{2}\) it lies outside the parabola and the next iterate lies to the right of \(x = \frac{1}{2}\). This proves the first assertion.

When \(d(z_{n+1}, w_1) = R < \infty\) we need to show for the proper choice of \(a > 1\) that \(|z_n - (\frac{1}{2} + h)| > r\). Changing the signs of \(H\) and \(h\) in (1) and rearranging we get

\[
|z_n - (\frac{1}{2} + h)|^2 - r^2 \\
= 2(R \cos \theta + H)[2R^2(R \cos \theta + H + 1) - (h - H)(2R \cos \theta + 1)].
\]

The first factor is positive and \(R \cos \theta + H + 1 > 2R \cos \theta + 1\) so all we require is \(2R^2 \geq h - H\). Now \((2R^2 + H)^2 - h^2 = R^2(4R^2 + 4H + 1 - a^2)\). Since \(R \leq H\) the expression is nonnegative for \(a = 2R + 1\) which implies \(r = R(2R + 1)\) so \(R < (r/2)^{1/2}\). By the quadratic formula \(1/R \geq [1 + (1 + 8r)^{1/2}]/2r\) which yields the next two results. We get the final result with \(a^2 = 1 + 4k < 1 + 4h\).
If $c > \frac{1}{3}$ then $w_1 = \frac{1}{3} + ki$ and $w_2 = \frac{1}{3} - ki$. The centers of the circles of the pencil lie on $x = \frac{1}{3}$ and the circles with center $\frac{1}{3} \pm Hi$ have radius $R$ where $R^2 + k^2 = H^2$. The real axis is the radical axis. For each $z$ not on the radical axis there is precisely one circle $C(z)$ of the pencil passing through $z$. We define $d(z, w_i)$ to be the radius of $C(z)$ if $\text{Im}(z) > 0$ and $\infty$ otherwise and $d(z, w_2)$ to be the radius of $C(z)$ if $\text{Im}(z) < 0$ and $\infty$ otherwise.

**Theorem 3.** For $c > \frac{1}{3}$, if $z_0$ is real then $d(z_n, w_1)$ is eventually finite. In fact, for a fixed $z_0 > \frac{1}{2}$ and $c$ near $\frac{1}{3}$, the number of iterations required is asymptotic to $\pi/k$. If $0 < d(z_n, w_i) < \infty$, then

$$d(z_{n+1}, w_i) \leq (1 + k^2)^{-1/2}d(z_n, w_i), \quad i = 1, 2.$$ 

**Proof.** If $z_{n+1}$ is real, then $z_{n+1} = (z_n - c)^{1/2} \leq (z_n - c) + \frac{1}{2} = z_n - (c - \frac{1}{2}).$ Thus some iterate is less than $c$ and the next lies in the first quadrant. If $z_0 < 2^{2n}$ then clearly $z_n < 2$ for some $n \leq m$. To estimate the number $N$ of steps required for $z$ to move from 2 to $c$ for small values of $k$, observe that

$$z_{n+1} - z_n = z_{n+1} - (z_n^2 + c) = - (z_{n+1} - \frac{1}{2})^2 - (c - \frac{1}{2}),$$

so if we let $z_n = \frac{1}{2} - kx_n$, $x$ ranges from $-3/2k$ to $(\frac{1}{2} - k^2)/k$ and $x_{n+1} - x_n = (x_{n+1} + 1)k$. Let $p$ be a positive integer. If $x_n \geq 2^p$ then $x_{n+1} - x_n > 2^{2p}p^2k$ so that the number of steps for $x$ to move from $2^p$ to $2^{j+1}p$ is less than

$$(2^{j+1}p - 2^j p)/2^{2j}p^2k + 1 = 1/2^jp + 1, \quad j = 0, 1, \ldots.$$ 

Thus the total number of steps required to move from $p$ to $(\frac{1}{2} - k^2)/k$ is less than $2/pk + \log_2[(\frac{1}{2} - k^2)/pk] < 3/pk$. A similar argument shows that the number of steps from $-3/2k$ to $-p$ is also bounded by $3/pk$. Now take $x_0 = -p$ and $x_M = p$, i.e. $M$ is the number of steps from $-p$ to $p$. Then

$$\sum_{n=0}^{M-1} \frac{x_{n+1} - x_n}{x_{n+1}^2 + 1} = Mk.$$ 

The left side is a Riemann sum for $\int_p^{-p} dx/(x^2 + 1)$ and since $x_{n+1} - x_n \leq (p^2 + 1)k$, the norm approaches zero with $k$. Clearly,

$$2 \tan^{-1} p < \lim_{k \to 0} Mk < 2 \tan^{-1} p + \frac{6}{p} < \pi + \frac{6}{p}.$$ 

Since $p$ can be arbitrarily large we have the result. Computer calculations show that if $c = 0.250001$ i.e. $k = 0.001$ and $z_0 = 2$, then $z_{3139} < c < z_{3138}$.

Suppose now $d(z_{n+1}, w_1) = R < \infty$, i.e.

$$z_{n+1} = \left(\frac{1}{3} + Hi\right) + R \cos \theta + iR \sin \theta = \frac{1}{3} + R \cos \theta + iv,$$
where \( v = H + R \sin \theta > 0 \). We shall show that \( d(z_n, w_1) \geq (1 + k^2)^{1/2}R = r \) i.e. that \( |z_n - (\frac{1}{2} + hi)| \geq r \) where \( h^2 = r^2 + k^2 \). In fact,

\[
|z_n - (\frac{1}{2} + hi)|^2 - r^2 = |z_{n+1} + c - (\frac{1}{2} + hi)|^2 - r^2
\]

\[
= |(\frac{1}{2} + R \cos \theta + iv)^2 + (H^2 - R^2 + 1) - (\frac{1}{2} + hi)|^2 - r^2
\]

\[
= |R(\cos \theta - 2v \sin \theta) + [R(\sin \theta + 2v \cos \theta) + H - h]|^2 - r^2,
\]

substituting for \( v^2 \),

\[
= R^2(\cos \theta - 2v \sin \theta)^2 + [R(\sin \theta + 2v \cos \theta) + H - h]^2 - r^2
\]

\[
= 2v[2vR^2 - (1 + 2R \cos \theta)(h - H)],
\]

using \( h^2 - r^2 = H^2 - R^2 \). The first factor is positive. The second rearranges to

\[
2HR^2 - (h - H) + 2R[R^2 \sin \theta - (h - H) \cos \theta]
\]

\[
\geq 2HR^2 - (h - H) - 2R[R^2 + (h - H)^2]^{1/2},
\]

via the Schwartz inequality,

\[
= R^2(h + H)^{-1}[2H(h + H) - k^2 - 2R[(h + H)^2 + k^4]^{1/2}]
\]

since \( h - H = (h^2 - H^2)/(h + H) = k^2R^2/(h + H) \). Further, \( 2H(h + H) - k^2 \geq 3k^2 > 0 \) so positivity follows from

\[
[2H(h + H) - k^2] - 4R^2[(h + H)^2 + k^4]
\]

\[
= k^2(4k^2R^2 + 4Hh + 5k^2) > 0.
\]

If \( d(z_0, w_2) < \infty \), the convergence of \( z \) to \( w_2 \) is identical to the convergence of \( z \) to \( w_1 \).

**Reference**