BERNOULLI SHIFTS ARE DETERMINED
BY THEIR FACTOR ALGEBRAS

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We shall show that, if $S$ is an invertible measure-preserving trans-
mation of the unit interval with the same factor algebras as a Bernoulli shift
$T$, then $S$ is isomorphic to $T$. This answers affirmatively a conjecture by
G.-C. Rota in 1968.

Definitions. A transformation $T$ is an invertible measure-preserving
transformation of the unit interval $X$ with Lebesgue sets $\Sigma$ and Lebesgue
measure $\mu$. $T$ is called ergodic if there are no $T$-invariant sets $A \in \Sigma$ with
$0 < \mu(A) < 1$. A factor algebra of $T$ is a complete $\sigma$-algebra $\Sigma_1 \subseteq \Sigma$ such
that $T\Sigma_1 = T^{-1}\Sigma_1 = \Sigma_1$. A partition $P$ is a countable disjoint collection of
measureable sets whose union is $X$. The join of $P$ and $Q$ is $P \vee Q =
\{ p \cap q | p \in P, q \in Q \}$ and $\bigvee_n T^i P = T^k P \vee T^{k+1} P \vee \cdots \vee T^n P$. Also, $\bigvee_{-\infty}^\infty T^i P$
denotes the smallest factor algebra containing $P$. $P$ is a generator for $T$
if $\bigvee_{-\infty}^\infty T^i P = \Sigma$ and $\{ T^i P \}$ is independent if, for each $n > 0$, $P$
is independent of $\bigvee_{1}^n T^i P$. $T$ is a Bernoulli shift if it has a generator $P$
such that $\{ T^i P \}$ is independent.

The entropy $H(P) = - \sum_{p \in P} \mu(p) \log \mu(p)$, the relative entropy

$$H(T, P) = \lim_{n \to \infty} H\left( \bigvee_{1}^n T^i P \right)$$

and $H(T) = \sup P H(T, P)$.

We have

1. $H(T, P) \leq H(P)$ with equality iff $\{ T^i P \}$ is independent.
2. If $P$ is a generator for $T$, then $H(T) = H(T, P)$.
3. If $T$ is ergodic and $\epsilon > 0$, there is a generator $P$ such that $H(P) \leq
H(T) + \epsilon$.

The reader is referred to [3] for details about entropy. The result (3)
is due to Rohlin and is the primary tool in our proof of the following
theorem.

Theorem. If $S$ has the same factor algebras as a Bernoulli shift $T$, then
$S$ is isomorphic to $T$.
Proof. We begin by making two elementary observations:

4. $S$ is ergodic.
5. $\bigvee_{-\infty}^{\infty} T^i P = \bigvee_{-\infty}^{\infty} S^i P$ for any $P$.

To prove (4), note that, if $SA=A$, then $\{\Phi, A, X-A, X\}$ is a factor algebra of $S$, hence of $T$. Since $T^2$ is ergodic, this implies that $\mu(A)$ is 0 or 1. The result (5) follows from the fact that $P$ is contained in the factor algebra $\bigvee_{-\infty}^{\infty} T^i P$ of $T$, hence so is $\bigvee_{-\infty}^{\infty} S^i P$.

We now prove

6. $H(S) = H(T)$.

First use (3) to choose a generator $P$ for $S$ such that $H(P) \leq H(S) + \varepsilon$. $P$ must be a generator for $T$, from (5), and (1) and (2) give

$$H(T) \leq H(P) \leq H(S) + \varepsilon.$$ 

Now interchange the roles of $S$ and $T$ to obtain (6).

To complete the proof of the theorem, choose a generator $P$ for $T$ such that $\{T^i P\}$ is independent. We then have $H(T) = H(T, P) = H(P)$, and $P$ is a generator for $S$. Therefore,

$$H(S) = H(S, P) \leq H(P) = H(T) = H(S),$$

so (1) implies that $\{S^i P\}$ is independent.

Remarks. 1. The result (6) only uses the fact that $T^2$ is ergodic and can be easily proved assuming only that $T$ is ergodic.

2. Any two irrational rotations of the unit circle have the same factor algebras [1].

3. If $T$ is a $K$-automorphism [3], then $S$ must also be a $K$-automorphism. However, since there is a $K$-automorphism $T$ which is not isomorphic to $T^{-1}$ (see [2]), the theorem is false for $K$-automorphisms.

References

